ABELIAN MONOID ON NEUTROSOPHIC FUZZY MATRICES USING LUKASIEWICZ DISJUNCTION AND CONJUNCTION OPERATOR

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Abstract
In this article, we study some of the properties of disjunction and conjunction from Lukasiewicz's type over Neutrosophic fuzzy sets and fuzzy matrices. Nowadays many fields facing the problem in indeterministic situation such as business, medical diagnosis, networking etc., We describe about max,min,min and min,max,max using neutrosophic fuzzy matrices and abelian Monoid on Lukasiewicz disjunction and conjuction are expounded Moreover, the connection between these concepts is established.
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1. Introduction

Intuitionistic Fuzzy Matrix (IFM) was raised by Atanossov.KT(1986). Samarandache(2005) extended many ideas in Neutrosophic Fuzzy Matrices (NFM). Elements of NFM is in the form of triplets . Each elements of neutrosophic fuzzy set is belong to [0,1]. They are fair one, unfair and indeterministic. Sum of these three should lies between 0 and 3. Khan SK et al. acquired the troop of intuitionistic fuzzy matrix. Pal M, Shyamal AK (2002)explained about Notes on Intuitionistic fuzzy sets. Kim, K. H. and Roush, R.W(1980) deals with generalized fuzzy matrices. The method of Lukasiewicz was brought by T.Muthuraji, S.Sriram(2015). D. Venkatesan and S.Sriram(2019) established Lukaseiwicz disjunction and conjunction operations over pythogorean fuzzy matrices. Vanmathi.N et al. brought up the ideas over split up of NFMs. Murugadas.P, Balasubramaniyan.K (2019) described the concepts of decomposition of Neutrosophic fuzzy matrices. Atanossov K, Tcvetkok R(2010)tells of on Lukaseiwicz intuitionistic fuzzy disjunction and conjunction. Lalitha. K, Buvaneswari. N raised the ideas of few equalities concatenated with intuititionistic fuzzy matrices. Muthuraji et al. (2016) obtained a decomposition of intuitionistic fuzzy matrices. Sriram and Boobalan (2016) studied the properties of algebraic sum and algebraic product of intuitionistic fuzzy matrices and prove that the set of all intuitionistic fuzzy matrices form a commutative monoid using Implication Operator. and established in neutrosophic fuzzy matrix.These two operations are proved under commutative monoid in NFM. In chapter 2 we mentioned the prelims. In

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chapter 3 explicate about Lukasiewicz disjunction and conjunction in NFM. We dispute some concepts of Lukasiewicz disjunction and Lukasiewicz conjunction.

2. Preliminaries

Definition 2.1[6].

Let Γ be a neutrosophic fuzzy set on the universal set X.It is defined as $\Gamma = (\langle \gamma_A, \gamma_I, \gamma_F \rangle)$. such that $\gamma_A, \gamma_I, \gamma_F : X \to (0, 1^+)$Where $0 \le \gamma_A + \gamma_I + \gamma_F \le 3$. Where $\gamma_A, \gamma_I, \gamma_F$ denotes the truthness, indeterminancy, falsity respectively.

Definition 2.2[6].

Each element of Neutrosophic fuzzy matrix is defined as $\Gamma = \left(\langle \gamma_{A_{dk}}, \gamma_{I_{dk}}, \gamma_{F_{dk}} \rangle\right)$. Whose values of $\gamma_{A_{dk}}, \gamma_{I_{dk}}, \gamma_{F_{dk}}$ are non-negative real numbers which belongs to [0,1] and fulfilling the state $0 \le \gamma_{A_{dk}} + \gamma_{I_{dk}} + \gamma_{F_{dk}} \le 3$ for every d, k.

Definition 2.3.

We introduce Lukasiewicz conjunction and disjunction form using neutrosophic fuzzy sets.

$\Gamma = \left(\langle \gamma_{A_{dk}}, \gamma_{I_{dk}}, \gamma_{F_{dk}} \rangle\right)$

$\Psi = \left(\langle \psi_{A_{dk}}, \psi_{I_{dk}}, \psi_{F_{dk}} \rangle\right)$.

$\Gamma \vee_L^* \Psi = \left(\langle \max\left(0, \gamma_{A_{dk}} + \psi_{A_{dk}} - 1\right), \min\left(1, \gamma_{I_{dk}} + \psi_{I_{dk}}\right), \min\left(1, \gamma_{F_{dk}} + \psi_{F_{dk}}\right) \rangle\right)$

$\Gamma \wedge_L^* \Psi = \left(\langle \min\left(1, \gamma_{A_{dk}} + \psi_{A_{dk}}\right), \max\left(0, \gamma_{I_{dk}} + \psi_{I_{dk}} - 1\right), \max\left(0, \gamma_{F_{dk}} + \psi_{F_{dk}} - 1\right) \rangle\right)$

Definition 2.4[6].

Let $\Gamma, \Psi \in \mathcal{F}_{mn}$

(i) If $\Gamma \ge \Psi$ iff $\gamma_{A_{dk}} \ge \psi_{A_{dk}}$; $\Upsilon_{I_{dk}} \le \psi_{I_{dk}}$; $\gamma_{F_{dk}} \le \psi_{F_{dk}}$.

(ii) $\left(\max\left(\gamma_{A_{dk}}, \psi_{A_{dk}}\right), \min\left(\gamma_{I_{dk}}, \psi_{I_{dk}}\right), \min\left(\gamma_{F_{dk}}, \psi_{F_{dk}}\right)\right) = \Gamma \vee \Psi$

(iii) $\left(\min\left(\gamma_{A_{dk}}, \psi_{A_{dk}}\right), \max\left(\gamma_{I_{dk}}, \psi_{I_{dk}}\right), \max\left(\gamma_{F_{dk}}, \psi_{F_{dk}}\right)\right) = \Gamma \wedge \Psi$

(iv) $\left(\langle \gamma_{F_{dk}}, 1 - \gamma_{I_{dk}}, \gamma_{A_{dk}} \rangle\right) = \Gamma^c$ (or) $\left(\langle 1 - \gamma_{F_{dk}}, 1 - \gamma_{I_{dk}}, \gamma_{A_{dk}} \rangle\right) = \Gamma^c$ or $\left(\langle \gamma_{F_{dk}}, \gamma_{I_{dk}}, \gamma_{A_{dk}} \rangle\right) = \Gamma^c$

3. Some results on NFMs

Definition 3.1[6].

$\Gamma = \left(\langle \gamma_{A_{dk}}, \gamma_{I_{dk}}, \gamma_{F_{dk}} \rangle\right)$ is said to be neutrosophic fuzzy matrix. We define

(i) Γ is symmetric iff $\Gamma = \Gamma^T$ i.e., $\left(\langle \gamma_{A_{dk}}, \gamma_{I_{dk}}, \gamma_{F_{dk}} \rangle\right) = \left(\langle \gamma_{A_{kd}}, \Upsilon_{I_{kd}}, \gamma_{F_{kd}} \rangle\right)$.

(ii) Γ is reflexive iff $\left(\langle \gamma_{A_{dd}}, \gamma_{I_{dd}}, \gamma_{F_{dd}} \rangle\right) = (\langle 1,0,0 \rangle)$

(iii) Γ is irreflexive iff $\left(\langle \gamma_{A_{dd}}, \gamma_{I_{dd}}, \gamma_{F_{dd}} \rangle\right) = (\langle 0,1,1 \rangle)$

(iv) Γ is weakly irreflexive iff $\left(\langle \gamma_{A_{dd}}, \gamma_{I_{dd}}, \gamma_{F_{dd}} \rangle\right) \ge \left(\langle \gamma_{A_{dk}}, \gamma_{I_{dk}}, \gamma_{F_{dk}} \rangle\right) \forall d, k$.

(v) Γ is transitive $\Gamma^2 \le \Gamma$

(vi) Γ is idempotent iff $\Gamma^2 = \Gamma$

(vii) Γ is nearly irreflexive if $\left(\langle \gamma_{A_{dd}}, \gamma_{I_{dd}}, \gamma_{F_{dd}} \rangle\right) \le \left(\langle \gamma_{A_{dk}}, \gamma_{I_{dk}}, \gamma_{F_{dk}} \rangle\right) \forall d, k$.

Proposition 3.2.

Let Γ and Ψ be any two NFMs. The following conditions are satisfied.
(i) If Γ and Ψ are reflexive then $\Gamma \vee_L^* \Psi$ and $\Gamma \wedge_L^* \Psi$ are reflexive.
(ii) If Γ and Ψ are irreflexive then $\Gamma \vee_L^* \Psi$ and $\Gamma \wedge_L^* \Psi$ are irreflexive.
(iii) If Γ and Ψ are symmetric then $\Gamma \vee_L^* \Psi$ and $\Gamma \wedge_L^* \Psi$ are symmetric.
(iv) If Γ and Ψ are nearly irreflexive then $\Gamma \vee_L^* \Psi$ and $\Gamma \wedge_L^* \Psi$ are nearly irreflexive.

Proof:

By the definition of reflexive

$$\Gamma = \left(\langle \gamma_{A_{dd}}, \gamma_{I_{dd}}, \gamma_{F_{dd}} \rangle\right) = (\langle 1,0,0 \rangle)$$

$$\Psi = \left(\langle \psi_{A_{dd}}, \psi_{I_{dd}}, \psi_{F_{dd}} \rangle\right) = (\langle 1,0,0 \rangle)$$

$$\begin{aligned} (i)\,\Gamma \vee_L^* \Psi &= \left(\langle \max\left(0, \gamma_{A_{dk}} + \psi_{A_{dk}} - 1\right), \min\left(1, \gamma_{I_{dk}} + \psi_{I_{dk}}\right), \min\left(1, \gamma_{F_{dk}} + \psi_{F_{dk}}\right) \rangle\right) \\ &= (\langle \max(0,1+1-1), \min(1,0+0), \min(1,0+0) \rangle) \\ &= (\langle 1,0,0 \rangle) \end{aligned}$$

$$\begin{aligned} (ii)\,\Gamma \wedge_L^* \Psi &= \\ &\left(\langle \min\left(1, \gamma_{A_{dk}} + \psi_{A_{dk}}\right), \max\left(0, \gamma_{I_{dk}} + \psi_{I_{dk}} - 1\right), \max\left(0, \gamma_{F_{dk}} + \psi_{F_{dk}} - 1\right) \rangle\right) \\ &= (\langle \min(1,1), \max(0,0+0-1), \max(0,0+0-1) \rangle) \\ &= (\langle 1,0,0 \rangle) \end{aligned}$$

Therefore (i) and (ii) are proved.

(iii) Given Γ and Ψ are symmetric.

$$\left(\langle \gamma_{A_{dk}}, \gamma_{I_{dk}}, \gamma_{F_{dk}} \rangle\right) = \left(\langle \gamma_{A_{kd}}, \Upsilon_{I_{kd}}, \gamma_{F_{kd}} \rangle\right)$$

$$\left(\langle \psi_{A_{dk}}, \psi_{I_{dk}}, \psi_{F_{dk}} \rangle\right) = \left(\langle \psi_{A_{kd}}, \psi_{I_{kd}}, \psi_{F_{kd}} \rangle\right)$$

Let $\Upsilon = \Gamma \vee_L^* \Psi$ and $T = \Gamma \wedge_L^* \Psi$

Where $\Upsilon = \left(\langle \upsilon_{A_{dk}}, \upsilon_{I_{dk}}, \upsilon_{F_{dk}} \rangle\right)$

$T = \left(\langle \tau_{A_{dk}}, \tau_{I_{dk}}, \tau_{F_{dk}} \rangle\right)$

$$\begin{aligned} &(< \upsilon_{A_{dk}}, \upsilon_{I_{dk}}, \upsilon_{F_{dk}} >) \\ &\quad = \left(\langle \max\left(0, \gamma_{A_{dk}} + \psi_{A_{dk}} - 1\right), \min\left(1, \gamma_{I_{dk}} + \psi_{I_{dk}}\right), \min\left(1, \gamma_{F_{dk}} + \psi_{F_{dk}}\right) \rangle\right) \end{aligned}$$

Case i)

If $\gamma_{A_{dk}} + \psi_{A_{dk}} \geq 1$ then $\gamma_{A_{dk}} + \psi_{A_{dk}} - 1 \geq 0$

Therefore $\max\left(0, \gamma_{A_{dk}} + \psi_{A_{dk}} - 1\right) = \gamma_{A_{dk}} + \psi_{A_{dk}} - 1$

If $\gamma_{I_{dk}} + \psi_{I_{dk}} \geq 1$ and $\gamma_{F_{dk}} + \psi_{F_{dk}} \geq 1$ then, $\min\left(1, \gamma_{I_{dk}} + \psi_{I_{dk}}\right) = 1$

And $\min\left(1, \gamma_{F_{dk}} + \psi_{F_{dk}}\right) = 1$

Then
$$\begin{aligned} \left(\langle \upsilon_{A_{dk}}, \upsilon_{I_{dk}}, \upsilon_{F_{dk}} \rangle\right) &= (\gamma_{A_{dk}} + \psi_{A_{dk}} - 1,1,1) \\ &= \left(\gamma_{A_{kd}} + \psi_{A_{kd}} - 1 - 1,1,1\right) \qquad (3.1) \\ \left(\langle \upsilon_{A_{dk}}, \upsilon_{I_{dk}}, \upsilon_{F_{dk}} \rangle\right) &= \left(\langle \upsilon_{A_{kd}}, \upsilon_{I_{kd}}, \upsilon_{F_{kd}} \rangle\right) \end{aligned}$$

Case 1.1)

If $\gamma_{A_{dk}} + \psi_{A_{dk}} \geq 1$ then $\gamma_{A_{dk}} + \psi_{A_{dk}} - 1 \geq 0$

And $\gamma_{I_{dk}} + \psi_{I_{dk}} < 1$, $\gamma_{F_{dk}} + \psi_{F_{dk}} < 1$

$$\left(\langle \upsilon_{A_{dk}}, \upsilon_{I_{dk}}, \upsilon_{F_{dk}} \rangle\right) = (\gamma_{A_{dk}} + \psi_{A_{dk}} - 1, \gamma_{I_{dk}} + \psi_{I_{dk}}, \gamma_{F_{dk}} + \psi_{F_{dk}})$$

$$= \left(\gamma_{A_{kd}} + \psi_{A_{kd}} - 1, \gamma_{I_{kd}} + \psi_{I_{kd}}, \gamma_{F_{kd}} + \psi_{F_{kd}}\right)$$
$$= \left(< \upsilon_{A_{kd}}, \upsilon_{I_{kd}}, \upsilon_{F_{kd}} >\right) \quad (3.2)$$

Case 2)

If$\gamma_{A_{dk}} + \psi_{A_{dk}} < 1$ then $\gamma_{A_{dk}} + \psi_{A_{dk}} - 1 < 0$

And $\gamma_{I_{dk}} + \psi_{I_{dk}} > 1$, $\gamma_{F_{dk}} + \psi_{F_{dk}} > 1$

$$\left(\langle \upsilon_{A_{dk}}, \upsilon_{I_{dk}}, \upsilon_{F_{dk}} \rangle\right) = (0,1,1)$$
$$= \left(\langle \upsilon_{A_{kd}}, \upsilon_{I_{kd}}, \upsilon_{F_{kd}} \rangle\right)$$

(3.3)

Case 2.1)

If $\gamma_{A_{dk}} + \psi_{A_{dk}} < 1$ then $\gamma_{A_{dk}} + \psi_{A_{dk}} - 1 < 0$

And $\gamma_{I_{dk}} + \psi_{I_{dk}} < 1$, $\gamma_{F_{dk}} + \psi_{F_{dk}} < 1$

$$\left(\langle \upsilon_{A_{dk}}, \upsilon_{I_{dk}}, \upsilon_{F_{dk}} \rangle\right) = \left(0, \gamma_{I_{dk}} + \psi_{I_{dk}}, \gamma_{F_{dk}} + \psi_{F_{dk}}\right)$$
$$= \left(0, \gamma_{I_{kd}} + \psi_{I_{kd}}, \gamma_{F_{kd}} + \psi_{F_{kd}}\right) \quad (3.4)$$

From (2),(3),(4),(5)

$$\left(\langle \zeta_{t_{rs}}, \vartheta_{t_{rs}}, \xi_{t_{rs}} \rangle\right) = \left(\langle \zeta_{t_{sr}}, \vartheta_{t_{sr}}, \xi_{t_{sr}} \rangle\right)$$

$\Rightarrow \Gamma \vee_L^* \Psi$ is symmetric

Similarly, we can prove $T = \Gamma \wedge_L^* \Psi$

$\Gamma \wedge_L^* \Psi$ is symmetric.

(iv) Since Γ and Ψ are nearly irreflexive

$$\left(\langle \gamma_{A_{dd}}, \gamma_{I_{dd}}, \gamma_{F_{dd}} \rangle\right) \leq \left(\langle \gamma_{A_{dk}}, \gamma_{I_{dk}}, \gamma_{F_{dk}} \rangle\right)$$
$$\left(\langle \psi_{A_{dd}}, \psi_{I_{dd}}, \psi_{F_{dd}} \rangle\right) \leq \left(\langle \psi_{A_{dk}}, \psi_{I_{dk}}, \psi_{F_{dk}} \rangle\right)$$

Let $\Gamma \vee_L^* \Psi$ be $\left(\langle \upsilon_{A_{dk}}, \upsilon_{I_{dk}}, \upsilon_{F_{dk}} \rangle\right)$

$$\left(\langle \upsilon_{A_{dk}}, \upsilon_{I_{dk}}, \upsilon_{F_{dk}} \rangle\right) =$$
$$\left(\langle \max\left(0, \gamma_{A_{dk}} + \psi_{A_{dk}} - 1\right), \min\left(1, \gamma_{I_{dk}} + \psi_{I_{dk}}\right), \min\left(1, \gamma_{F_{dk}} + \psi_{F_{dk}}\right) \rangle\right) \quad (3.5)$$

$$\left(\langle \upsilon_{A_{dd}}, \upsilon_{I_{dd}}, \upsilon_{F_{dd}} \rangle\right)$$
$$= \left(\langle \max\left(0, \gamma_{A_{dd}} + \psi_{A_{dd}} - 1\right), \min\left(1, \gamma_{I_{dd}} + \psi_{I_{dd}}\right), \min\left(1, \gamma_{F_{dd}} + \psi_{F_{dd}}\right) \rangle\right)$$

$\gamma_{A_{dd}} \leq \gamma_{A_{dk}}$ and $\psi_{A_{dd}} \leq \psi_{A_{dk}}$

$$\max\left(0, \gamma_{A_{dd}} + \psi_{A_{dd}} - 1\right) \leq \max\left(0, \gamma_{A_{dk}} + \psi_{A_{dk}} - 1\right) \quad (3.6)$$

Similarly,

$$\min\left(1, \gamma_{I_{dd}} + \psi_{I_{dd}}\right) \geq \min\left(1, \gamma_{I_{dk}} + \psi_{I_{dk}}\right) \quad (3.7)$$
$$\min\left(1, \gamma_{F_{dd}} + \psi_{F_{dd}}\right) \geq \min\left(1, \gamma_{F_{dk}} + \psi_{F_{dk}}\right) \quad (3.8)$$

From (3.6),(3.7),(3.7=3.8)

$$\left(\langle \upsilon_{A_{dd}}, \upsilon_{I_{dd}}, \upsilon_{F_{dd}} \rangle\right) \leq \left(\langle \upsilon_{A_{dk}}, \upsilon_{I_{dk}}, \upsilon_{F_{dk}} \rangle\right)$$

$\Gamma \vee_L^* \Psi$ is nearly irreflexive, $\Gamma \wedge_L^* \Psi$ is nearly irreflexive.

Proposition 3.3.

Let Γ be NFM, $\Gamma \in \mathcal{F}_{mn}$

(i) $I_n \wedge_L^* (\Gamma \vee_L^* \Gamma^T)$ is symmetric

(ii) $I_n \wedge_L^* (\Gamma \vee_L^* \Gamma^T)$ is reflexive

(iii) $I_n \wedge_L^* (\Gamma \vee_L^* \Gamma^T) = I_n \vee (\Gamma \vee_L^* \Gamma^T)$

Proof:

(i) Let $\Gamma = (\langle \gamma_{A_{dk}}, \gamma_{I_{dk}}, \gamma_{F_{dk}} \rangle)$

$\Gamma^T = (\langle \gamma_{A_{kd}}, \gamma_{I_{kd}}, \gamma_{F_{kd}} \rangle)$.

$$(\langle \max(0, \gamma_{A_{dk}} + \gamma_{A_{kd}} - 1), \min(1, \gamma_{I_{dk}} + \gamma_{I_{kd}}), \min(1, \gamma_{F_{kd}} + \gamma_{F_{kd}}) \rangle) = T$$

$$T = (\Gamma \vee_L^* \Gamma^T)$$

Where $T = (\langle \tau_{A_{dk}}, \tau_{I_{dk}}, \tau_{F_{dk}} \rangle)$

$$\Xi = (\langle \xi_{A_{dk}}, \xi_{I_{dk}}, \xi_{F_{dk}} \rangle) = I_n \wedge_L^* (\Gamma \vee_L^* \Gamma^T)$$

$$I_n \wedge_L^* (\Gamma \vee_L^* \Gamma^T) = \begin{cases} (\min(1,1 + \tau_{A_{dk}}), \max(0,0 + \tau_{I_{dk}} - 1), \max(0,0 + \tau_{F_{dk}} - 1)) & if\ d = k \\ (\min(1,0 + \tau_{A_{dk}}), \max(0,1 + \tau_{I_{dk}} - 1), \max(0,1 + \tau_{F_{dk}} - 1)) & if\ d \neq k \end{cases}$$

If $d = k$

$$(\langle \xi_{A_{dk}}, \xi_{I_{dk}}, \xi_{F_{dk}} \rangle) = (\langle \min(1,1 + \tau_{A_{dk}}), \max(0,0 + \tau_{I_{dk}} - 1), \max(0,0 + \tau_{F_{dk}} - 1) \rangle)$$

$$(\langle \xi_{A_{dk}}, \xi_{I_{dk}}, \xi_{F_{dk}} \rangle) = (\langle 1,0,0 \rangle) \quad (3.9)$$

If $d \neq k$

$$(\langle \xi_{A_{dk}}, \xi_{I_{dk}}, \xi_{F_{dk}} \rangle) = (\langle \min(1,0 + \tau_{A_{dk}}), \max(0,1 + \tau_{I_{dk}} - 1), \max(0,1 + \tau_{F_{dk}} - 1) \rangle)$$
$$= (\langle \min(1, \tau_{A_{dk}}), \max(0, \tau_{I_{dk}}), \max(0, \tau_{F_{dk}}) \rangle)$$

$$(\langle \xi_{A_{dk}}, \xi_{I_{dk}}, \xi_{F_{dk}} \rangle) = (\langle \xi_{A_{kd}}, \xi_{I_{kd}}, \xi_{F_{kd}} \rangle) \quad (3.10)$$

From (3.8),(3.9) ⇒ Ξ is symmetric.

(ii) $\Xi = (\langle \xi_{A_{dk}}, \xi_{I_{dk}}, \xi_{F_{dk}} \rangle) = I_n \wedge_L^* (\Gamma \vee_L^* \Gamma^T)$

$$I_n \wedge_L^* (\Gamma \vee_L^* \Gamma^T) = \begin{cases} (\min(1,1 + \tau_{A_{dk}}), \max(0,0 + \tau_{I_{dk}} - 1), \max(0,0 + \tau_{F_{dk}} - 1)) & if\ d = k \\ (\min(1,0 + \tau_{A_{dk}}), \max(0,1 + \tau_{I_{dk}} - 1), \max(0,1 + \tau_{F_{dk}} - 1)) & if\ d \neq k \end{cases}$$

If $d = k$

$$(\langle \xi_{A_{dk}}, \xi_{I_{dk}}, \xi_{F_{dk}} \rangle) = (\langle \min(1,1 + \tau_{A_{dk}}), \max(0,0 + \tau_{I_{dk}} - 1), \max(0,0 + \tau_{F_{dk}} - 1) \rangle)$$

$$(\langle \xi_{A_{dk}}, \xi_{I_{dk}}, \xi_{F_{dk}} \rangle) = (1,0,0)$$

Therefore Ξ is reflexive.

(iii) $I_n \wedge_L^* (\Gamma \vee_L^* \Gamma^T) = I_n \vee (\Gamma \vee_L^* \Gamma^T)$

If $k = d$, then $(\langle \xi_{A_{dk}}, \xi_{I_{dk}}, \xi_{F_{dk}} \rangle) = (1,0,0)$

If $k \neq d$, then $(\langle \xi_{A_{dk}}, \xi_{I_{dk}}, \xi_{F_{dk}} \rangle) = (\langle \tau_{A_{dk}}, \tau_{I_{dk}}, \tau_{F_{dk}} \rangle)$

$\Longrightarrow I_n \wedge_L^* (\Gamma \vee_L^* \Gamma^T) = I_n \vee (\Gamma \vee_L^* \Gamma^T)$

Proposition 3.4.
Let Γ, Ψ, Υ be three NFMs.
(i) $(\Gamma \vee_L^* \Psi) \vee_L^* \Upsilon = \Gamma \vee_L^* (\Psi \vee_L^* \Upsilon)$
(ii) $(\Gamma \wedge_L^* \Psi) \wedge_L^* \Upsilon = \Gamma \wedge_L^* (\Psi \wedge_L^* \Upsilon)$
Proof:
Let $(\Gamma \vee_L^* \Psi) = (\langle \tau_{A_{dk}}, \tau_{I_{dk}}, \tau_{F_{dk}} \rangle)$
$(\Psi \vee_L^* \Upsilon) = (\langle \xi_{A_{dk}}, \xi_{I_{dk}}, \xi_{F_{dk}} \rangle)$
$(\Gamma \vee_L^* \Psi) \vee_L^* \Upsilon = (\langle \sigma_{A_{dk}}, \sigma_{I_{dk}}, \sigma_{F_{dk}} \rangle)$
$\Gamma \vee_L^* (\Psi \vee_L^* \Upsilon) = (\langle \varkappa_{A_{dk}}, \varkappa_{I_{dk}}, \varkappa_{F_{dk}} \rangle)$

$$(\langle \tau_{A_{dk}}, \tau_{I_{dk}}, \tau_{F_{dk}} \rangle) = (\langle \max(0, \gamma_{A_{dk}} + \psi_{A_{dk}} - 1), \min(1, \gamma_{I_{dk}} + \psi_{I_{dk}}), \min(1, \gamma_{F_{dk}} + \psi_{F_{dk}}) \rangle)$$

$$(\langle \xi_{A_{dk}}, \xi_{I_{dk}}, \xi_{F_{dk}} \rangle) = (\langle \max(0, \psi_{A_{dk}} + \upsilon_{A_{dk}} - 1), \min(1, \psi_{I_{dk}} + \upsilon_{I_{dk}}), \min(1, \psi_{F_{dk}} + \upsilon_{F_{dk}}) \rangle)$$

$$(\langle \sigma_{A_{dk}}, \sigma_{I_{dk}}, \sigma_{F_{dk}} \rangle) = (\langle \max(0, \tau_{A_{dk}} + \upsilon_{A_{dk}} - 1), \min(1, \tau_{I_{dk}} + \upsilon_{I_{dk}}), \min(1, \tau_{F_{dk}} + \upsilon_{F_{dk}}) \rangle)$$

$$(\langle \varkappa_{A_{dk}}, \varkappa_{I_{dk}}, \varkappa_{F_{dk}} \rangle) = (\langle \max(0, \gamma_{A_{dk}} + \xi_{A_{dk}} - 1), \min(1, \gamma_{I_{dk}} + \xi_{I_{dk}}), \min(1, \gamma_{F_{dk}} + \xi_{F_{dk}}) \rangle)$$

To Show that $(\langle \sigma_{A_{dk}}, \sigma_{I_{dk}}, \sigma_{F_{dk}} \rangle) = (\langle \varkappa_{A_{dk}}, \varkappa_{I_{dk}}, \varkappa_{F_{dk}} \rangle)$

Case i) Suppose $\gamma_{A_{dk}} + \psi_{A_{dk}} < 1$; $\gamma_{A_{dk}} + \psi_{A_{dk}} - 1 \leq 0$
And $\gamma_{I_{dk}} + \psi_{I_{dk}} \geq 1$; $\gamma_{F_{dk}} + \psi_{F_{dk}} \geq 1$

$$(\langle \sigma_{A_{dk}}, \sigma_{I_{dk}}, \sigma_{F_{dk}} \rangle) = (\langle \max(0, \tau_{A_{dk}} + \upsilon_{A_{dk}} - 1), \min(1, \tau_{I_{dk}} + \upsilon_{I_{dk}}), \min(1, \tau_{F_{dk}} + \upsilon_{F_{dk}}) \rangle)$$

Suppose $\gamma_{A_{dk}} + \psi_{A_{dk}} + \upsilon_{A_{dk}} < 1$

$\gamma_{A_{dk}} + \psi_{A_{dk}} + \upsilon_{A_{dk}} - 1 < 0$; $\gamma_{I_{dk}} + \psi_{I_{dk}} + \upsilon_{I_{dk}} < 1$; $\gamma_{F_{dk}} + \psi_{F_{dk}} + \upsilon_{F_{dk}} < 1$

$$(\langle \sigma_{A_{dk}}, \sigma_{I_{dk}}, \sigma_{F_{dk}} \rangle) = (\max(0, \gamma_{A_{dk}} + \psi_{A_{dk}} + \upsilon_{A_{dk}} - 1 - 1), \min(1, \gamma_{I_{dk}} + \psi_{I_{dk}} + \upsilon_{I_{dk}}), \min(1, \gamma_{F_{dk}} + \psi_{F_{dk}} + \upsilon_{F_{dk}}))$$

$$= (\langle 0, \gamma_{I_{dk}} + \psi_{I_{dk}} + \upsilon_{I_{dk}}, \gamma_{F_{dk}} + \psi_{F_{dk}} + \upsilon_{F_{dk}} \rangle)$$
$$= (\langle \varkappa_{A_{dk}}, \varkappa_{I_{dk}}, \varkappa_{F_{dk}} \rangle)$$

1.1) Suppose $\gamma_{A_{dk}} + \psi_{A_{dk}} + \upsilon_{A_{dk}} < 1$

$\gamma_{I_{dk}} + \psi_{I_{dk}} + \upsilon_{I_{dk}} - 1 \leq 0$;
Either $\gamma_{I_{dk}} + \psi_{I_{dk}} + \upsilon_{I_{dk}} \geq 1$ or $\gamma_{I_{dk}} + \psi_{I_{dk}} + \upsilon_{I_{dk}} < 1$;

Either $\gamma_{F_{dk}} + \psi_{F_{dk}} + \upsilon_{F_{dk}} \leq 1$ or $\gamma_{F_{dk}} + \psi_{F_{dk}} + \upsilon_{F_{dk}} \leq 1$

$$(\langle \sigma_{A_{dk}}, \sigma_{I_{dk}}, \sigma_{F_{dk}} \rangle) = (< \max(0, \gamma_{A_{dk}} + \psi_{A_{dk}} + \upsilon_{A_{dk}} - 1 - 1), \min(1, \gamma_{I_{dk}} + \psi_{I_{dk}} + \upsilon_{I_{dk}}), \min(1, \gamma_{F_{dk}} + \psi_{F_{dk}} + \upsilon_{F_{dk}}) >)$$

$$= (\langle 0,1,1 \rangle)$$
$$= (\langle \varkappa_{A_{dk}}, \varkappa_{I_{dk}}, \varkappa_{F_{dk}} \rangle)$$

1.2) Suppose $\gamma_{A_{dk}} + \psi_{A_{dk}} + \upsilon_{A_{dk}} < 1$

$\gamma_{A_{dk}} + \psi_{A_{dk}} + \upsilon_{A_{dk}} - 1 \leq 0$;
$\gamma_{I_{dk}} + \psi_{I_{dk}} + \upsilon_{I_{dk}} \geq 1$
$\gamma_{F_{dk}} + \psi_{F_{dk}} + \upsilon_{F_{dk}} < 1$

$$(\langle \sigma_{A_{dk}}, \sigma_{I_{dk}}, \sigma_{F_{dk}} \rangle) = (< \max(0, \gamma_{A_{dk}} + \psi_{A_{dk}} + \upsilon_{A_{dk}} - 1 - 1), \min(1, \gamma_{I_{dk}} + \psi_{I_{dk}} + \upsilon_{I_{dk}}), \min(1, \gamma_{F_{dk}} + \psi_{F_{dk}} + \upsilon_{F_{dk}}) >)$$

$$= (\langle 0,1, \gamma_{F_{dk}} + \psi_{F_{dk}} + \upsilon_{F_{dk}} \rangle)$$
$$= (\langle \varkappa_{A_{dk}}, \varkappa_{I_{dk}}, \varkappa_{F_{dk}} \rangle)$$

1.3) If $\gamma_{A_{dk}} + \psi_{A_{dk}} + \upsilon_{A_{dk}} < 1$

$\gamma_{A_{dk}} + \psi_{A_{dk}} + \upsilon_{A_{dk}} - 1 \leq 0$;
$\gamma_{I_{dk}} + \psi_{I_{dk}} + \upsilon_{I_{dk}} < 1$
$\gamma_{F_{dk}} + \psi_{F_{dk}} + \upsilon_{F_{dk}} \geq 1$ then,

$$(\langle \sigma_{A_{dk}}, \sigma_{I_{dk}}, \sigma_{F_{dk}} \rangle) = (< \max(0, \gamma_{A_{dk}} + \psi_{A_{dk}} + \upsilon_{A_{dk}} - 1 - 1), \min(1, \gamma_{I_{dk}} + \psi_{I_{dk}} + \upsilon_{I_{dk}}), \min(1, \gamma_{F_{dk}} + \psi_{F_{dk}} + \upsilon_{F_{dk}}) >)$$

$$= (\langle 0, \gamma_{I_{dk}} + \psi_{I_{dk}} + \upsilon_{I_{dk}}, 1 \rangle)$$
$$= (\langle \varkappa_{A_{dk}}, \varkappa_{I_{dk}}, \varkappa_{F_{dk}} \rangle)$$

Case 2:

If $\gamma_{A_{dk}} + \psi_{A_{dk}} + \upsilon_{A_{dk}} - 1 > 0$

$\gamma_{I_{dk}} + \psi_{I_{dk}} + \upsilon_{I_{dk}} > 1$
$\gamma_{F_{dk}} + \psi_{F_{dk}} + \upsilon_{F_{dk}} > 1$ then,

$$(\langle \sigma_{A_{dk}}, \sigma_{I_{dk}}, \sigma_{F_{dk}} \rangle) = (< \max(0, \gamma_{A_{dk}} + \psi_{A_{dk}} + \upsilon_{A_{dk}} - 1), \min(1, \gamma_{I_{dk}} + \psi_{I_{dk}} + \upsilon_{I_{dk}}), \min(1, \gamma_{F_{dk}} + \psi_{F_{dk}} + \upsilon_{F_{dk}}) >)$$

$$= (\langle \gamma_{A_{dk}} + \psi_{A_{dk}} + \upsilon_{A_{dk}} - 1,1,1 \rangle)$$

$$= \left(\langle \varkappa_{A_{dk}}, \varkappa_{I_{dk}}, \varkappa_{F_{dk}} \rangle\right)$$

2.1) If $\gamma_{A_{dk}} + \psi_{A_{dk}} + \upsilon_{A_{dk}} - 1 > 0$

$\gamma_{I_{dk}} + \psi_{I_{dk}} + \upsilon_{I_{dk}} < 1$
$\gamma_{F_{dk}} + \psi_{F_{dk}} + \upsilon_{F_{dk}} < 1$ then,

$$\left(\langle \sigma_{A_{dk}}, \sigma_{I_{dk}}, \sigma_{F_{dk}} \rangle\right) = (< \max\left(0, \gamma_{A_{dk}} + \psi_{A_{dk}} + \upsilon_{A_{dk}} - 1\right), \min\left(1, \gamma_{I_{dk}} + \psi_{I_{dk}} + \upsilon_{I_{dk}}\right), \min\left(1, \gamma_{F_{dk}} + \psi_{F_{dk}} + \upsilon_{F_{dk}}\right) >)$$

$$= \left(\langle \gamma_{A_{dk}} + \psi_{A_{dk}} + \upsilon_{A_{dk}} - 1, \gamma_{I_{dk}} + \psi_{I_{dk}} + \upsilon_{I_{dk}}, \gamma_{F_{dk}} + \psi_{F_{dk}} + \upsilon_{F_{dk}} \rangle\right)$$
$$= \left(\langle \varkappa_{A_{dk}}, \varkappa_{I_{dk}}, \varkappa_{F_{dk}} \rangle\right)$$

2.2) If $\gamma_{A_{dk}} + \psi_{A_{dk}} + \upsilon_{A_{dk}} - 1 > 0$

$\gamma_{I_{dk}} + \psi_{I_{dk}} + \upsilon_{I_{dk}} > 1$
$\gamma_{F_{dk}} + \psi_{F_{dk}} + \upsilon_{F_{dk}} < 1$ then,

$$\left(\langle \sigma_{A_{dk}}, \sigma_{I_{dk}}, \sigma_{F_{dk}} \rangle\right) = (< \max\left(0, \gamma_{A_{dk}} + \psi_{A_{dk}} + \upsilon_{A_{dk}} - 1\right), \min\left(1, \gamma_{I_{dk}} + \psi_{I_{dk}} + \upsilon_{I_{dk}}\right), \min\left(1, \gamma_{F_{dk}} + \psi_{F_{dk}} + \upsilon_{F_{dk}}\right) >)$$

$$= \left(\langle \gamma_{A_{dk}} + \psi_{A_{dk}} + \upsilon_{A_{dk}} - 1, 1, \gamma_{F_{dk}} + \psi_{F_{dk}} + \upsilon_{F_{dk}} \rangle\right)$$
$$= \left(\langle \varkappa_{A_{dk}}, \varkappa_{I_{dk}}, \varkappa_{F_{dk}} \rangle\right)$$

Therefore, in all cases $\left(\langle \sigma_{A_{dk}}, \sigma_{I_{dk}}, \sigma_{F_{dk}} \rangle\right) = \left(\langle \varkappa_{A_{dk}}, \varkappa_{I_{dk}}, \varkappa_{F_{dk}} \rangle\right)$

$$(\varGamma \vee_L^* \varPsi) \vee_L^* \varUpsilon = \varGamma \vee_L^* (\varPsi \vee_L^* \varUpsilon)$$

*(ii)*Let $(\varGamma \wedge_L^* \varPsi) = \left(\langle \varsigma_{A_{dk}}, \varsigma_{I_{dk}}, \varsigma_{F_{dk}} \rangle\right)$

$$(\varPsi \wedge_L^* \varUpsilon) = \left(\langle \varpi_{A_{dk}}, \varpi_{I_{dk}}, \varpi_{F_{dk}} \rangle\right)$$
$$(\varGamma \wedge_L^* \varPsi) \wedge_L^* \varUpsilon = \left(\langle \nu_{A_{dk}}, \nu_{I_{dk}}, \nu_{F_{dk}} \rangle\right)$$
$$\varGamma \wedge_L^* (\varPsi \wedge_L^* \varUpsilon) = \left(\langle \lambda_{A_{dk}}, \lambda_{I_{dk}}, \lambda_{F_{dk}} \rangle\right)$$

$$\left(\langle \varsigma_{A_{dk}}, \varsigma_{I_{dk}}, \varsigma_{F_{dk}} \rangle\right)$$

$$\left(\langle \varpi_{A_{dk}}, \varpi_{I_{dk}}, \varpi_{F_{dk}} \rangle\right) = \left(\langle \min\left(1, \psi_{A_{dk}} + \upsilon_{A_{dk}}\right), \max\left(0, \psi_{I_{dk}} + \upsilon_{I_{dk}} - 1\right), \max\left(0, \psi_{F_{dk}} + \upsilon_{F_{dk}} - 1\right) \rangle\right)$$

$$\left(\langle \nu_{A_{dk}}, \nu_{I_{dk}}, \nu_{F_{dk}} \rangle\right) = \left(\langle \min\left(1, \varsigma_{A_{dk}} + \upsilon_{A_{dk}}\right), \max\left(0, \varsigma_{I_{dk}} + \upsilon_{I_{dk}} - 1\right), \max\left(0, \varsigma_{F_{dk}} + \upsilon_{F_{dk}} - 1\right) \rangle\right)$$

$$\left(\langle\lambda_{A_{dk}},\lambda_{I_{dk}},\lambda_{F_{dk}}\rangle\right) = \left(\langle\min\left(1,\gamma_{A_{dk}}+\varpi_{A_{dk}}\right),\max\left(0,\gamma_{I_{dk}}+\varpi_{I_{dk}}-1\right),\max\left(0,\gamma_{F_{dk}}+\varpi_{F_{dk}}-1\right)\rangle\right)$$

To Show that $\left(\langle\nu_{A_{dk}},\nu_{I_{dk}},\nu_{F_{dk}}\rangle\right) = \left(\langle\lambda_{A_{dk}},\lambda_{I_{dk}},\lambda_{F_{dk}}\rangle\right)$

Case i) Suppose $\gamma_{A_{dk}}+\psi_{A_{dk}} < 1$; $\gamma_{A_{dk}}+\psi_{A_{dk}}-1 \le 0$

And $\gamma_{I_{dk}}+\psi_{I_{dk}} \ge 1$; $\gamma_{F_{dk}}+\psi_{F_{dk}} \ge 1$

$$\left(\langle\nu_{A_{dk}},\nu_{I_{dk}},\nu_{F_{dk}}\rangle\right) = \left(\langle\min\left(1,\varsigma_{A_{dk}}+\upsilon_{A_{dk}}\right),\max\left(0,\varsigma_{I_{dk}}+\upsilon_{I_{dk}}-1\right),\max\left(0,\varsigma_{F_{dk}}+\upsilon_{F_{dk}}-1\right)\rangle\right)$$

Suppose $\gamma_{A_{dk}}+\psi_{A_{dk}}+\upsilon_{A_{dk}} < 1$

$\gamma_{A_{dk}}+\psi_{A_{dk}}+\upsilon_{A_{dk}} < 1$; $\gamma_{I_{dk}}+\psi_{I_{dk}}+\upsilon_{I_{dk}}-1 < 0$; $\gamma_{F_{dk}}+\psi_{F_{dk}}+\upsilon_{F_{dk}}-1 < 0$

$$\left(\langle\nu_{A_{dk}},\nu_{I_{dk}},\nu_{F_{dk}}\rangle\right) = \begin{pmatrix}\min\left(1,\gamma_{A_{dk}}+\psi_{A_{dk}}+\upsilon_{A_{dk}}\right),\max\left(0,\gamma_{I_{dk}}+\psi_{I_{dk}}+\upsilon_{I_{dk}}-1\right),\\ \max\left(0,\gamma_{F_{dk}}+\psi_{F_{dk}}+\upsilon_{F_{dk}}-1\right)\end{pmatrix}$$

$$= \left(\langle\gamma_{A_{dk}}+\psi_{A_{dk}}+\upsilon_{A_{dk}},0,0\rangle\right)$$
$$= \left(\langle\lambda_{A_{dk}},\lambda_{I_{dk}},\lambda_{F_{dk}}\rangle\right)$$

1.4) Suppose $\gamma_{A_{dk}}+\psi_{A_{dk}}+\upsilon_{A_{dk}} < 1$;

Either $\gamma_{I_{dk}}+\psi_{I_{dk}}+\upsilon_{I_{dk}} \ge 1$ or $\gamma_{I_{dk}}+\psi_{I_{dk}}+\upsilon_{I_{dk}} < 1$;
Either $\gamma_{F_{dk}}+\psi_{F_{dk}}+\upsilon_{F_{dk}} \ge 1$ or $\gamma_{F_{dk}}+\psi_{F_{dk}}+\upsilon_{F_{dk}} < 1$

$$\left(\langle\nu_{A_{dk}},\nu_{I_{dk}},\nu_{F_{dk}}\rangle\right) = \begin{pmatrix}\min\left(1,\gamma_{A_{dk}}+\psi_{A_{dk}}+\upsilon_{A_{dk}}\right),\max\left(0,\gamma_{I_{dk}}+\psi_{I_{dk}}+\upsilon_{I_{dk}}-1-1\right),\\ \max\left(0,\gamma_{F_{dk}}+\psi_{F_{dk}}+\upsilon_{F_{dk}}-1-1\right)\end{pmatrix}$$

$$= \left(\langle\gamma_{A_{dk}}+\psi_{A_{dk}}+\upsilon_{A_{dk}},1,1\rangle\right)$$
$$= \left(\langle\lambda_{A_{dk}},\lambda_{I_{dk}},\lambda_{F_{dk}}\rangle\right)$$

1.5) Suppose $\gamma_{A_{dk}}+\psi_{A_{dk}}+\upsilon_{A_{dk}} < 1$

$\gamma_{I_{dk}}+\psi_{I_{dk}}+\upsilon_{I_{dk}}-1 \ge 0$

$\gamma_{F_{dk}}+\psi_{F_{dk}}+\upsilon_{F_{dk}}-1-1 < 0$

$$\left(\langle\nu_{A_{dk}},\nu_{I_{dk}},\nu_{F_{dk}}\rangle\right) = \begin{pmatrix}\min\left(1,\gamma_{A_{dk}}+\psi_{A_{dk}}+\upsilon_{A_{dk}}\right),\max\left(0,\gamma_{I_{dk}}+\psi_{I_{dk}}+\upsilon_{I_{dk}}-1-1\right),\\ \max\left(0,\gamma_{F_{dk}}+\psi_{F_{dk}}+\upsilon_{F_{dk}}-1-1\right)\end{pmatrix}$$

$$= \left(\langle\gamma_{A_{dk}}+\psi_{A_{dk}}+\upsilon_{A_{dk}},\gamma_{I_{dk}}+\psi_{I_{dk}}+\upsilon_{I_{dk}}-1,0\rangle\right)$$

$$=\left(\langle\lambda_{A_{dk}},\lambda_{I_{dk}},\lambda_{F_{dk}}\rangle\right)$$

1.6) If $\gamma_{A_{dk}}+\psi_{A_{dk}}+\upsilon_{A_{dk}}<1$

$\gamma_{A_{dk}}+\psi_{A_{dk}}+\upsilon_{A_{dk}}\leq 0$;
$\gamma_{I_{dk}}+\psi_{I_{dk}}+\upsilon_{I_{dk}}-1<0$
$\gamma_{F_{dk}}+\psi_{F_{dk}}+\upsilon_{F_{dk}}-1\geq 0$ then,

$$\left(\langle\nu_{A_{dk}},\nu_{I_{dk}},\nu_{F_{dk}}\rangle\right)=\begin{pmatrix}\min\left(1,\gamma_{A_{dk}}+\psi_{A_{dk}}+\upsilon_{A_{dk}}\right),\max\left(0,\gamma_{I_{dk}}+\psi_{I_{dk}}+\upsilon_{I_{dk}}-1-1\right),\\ \max\left(0,\gamma_{F_{dk}}+\psi_{F_{dk}}+\upsilon_{F_{dk}}-1-1\right)\end{pmatrix}$$

$$=\left(\langle\gamma_{A_{dk}}+\psi_{A_{dk}}+\upsilon_{A_{dk}},0,\gamma_{F_{dk}}+\psi_{F_{dk}}+\upsilon_{F_{dk}}-1\rangle\right)$$
$$=\left(\langle\lambda_{A_{dk}},\lambda_{I_{dk}},\lambda_{F_{dk}}\rangle\right)$$

Case 2:

If $\gamma_{A_{dk}}+\psi_{A_{dk}}+\upsilon_{A_{dk}}>1$

$\gamma_{I_{dk}}+\psi_{I_{dk}}+\upsilon_{I_{dk}}-1>0$
$\gamma_{F_{dk}}+\psi_{F_{dk}}+\upsilon_{F_{dk}}-1>0$ then,

$$\left(\langle\nu_{A_{dk}},\nu_{I_{dk}},\nu_{F_{dk}}\rangle\right)=\begin{pmatrix}\min\left(1,\gamma_{A_{dk}}+\psi_{A_{dk}}+\upsilon_{A_{dk}}\right),\max\left(0,\gamma_{I_{dk}}+\psi_{I_{dk}}+\upsilon_{I_{dk}}-1-1\right),\\ \max\left(0,\gamma_{F_{dk}}+\psi_{F_{dk}}+\upsilon_{F_{dk}}-1-1\right)\end{pmatrix}$$
$$=\left(\langle 1,\gamma_{I_{dk}}+\psi_{I_{dk}}+\upsilon_{I_{dk}}-1,\gamma_{F_{dk}}+\psi_{F_{dk}}+\upsilon_{F_{dk}}-1\rangle\right)$$
$$=\left(\langle\lambda_{A_{dk}},\lambda_{I_{dk}},\lambda_{F_{dk}}\rangle\right)$$

2.1) If $\gamma_{A_{dk}}+\psi_{A_{dk}}+\upsilon_{A_{dk}}>1$

$\gamma_{I_{dk}}+\psi_{I_{dk}}+\upsilon_{I_{dk}}-1<0$
$\gamma_{F_{dk}}+\psi_{F_{dk}}+\upsilon_{F_{dk}}-1<0$ then,

$$\left(\langle\nu_{A_{dk}},\nu_{I_{dk}},\nu_{F_{dk}}\rangle\right)=\begin{pmatrix}\min\left(1,\gamma_{A_{dk}}+\psi_{A_{dk}}+\upsilon_{A_{dk}}\right),\max\left(0,\gamma_{I_{dk}}+\psi_{I_{dk}}+\upsilon_{I_{dk}}-1-1\right),\\ \max\left(0,\gamma_{F_{dk}}+\psi_{F_{dk}}+\upsilon_{F_{dk}}-1-1\right)\end{pmatrix}$$

$$=\left(\langle\gamma_{A_{dk}}+\psi_{A_{dk}}+\upsilon_{A_{dk}},\gamma_{I_{dk}}+\psi_{I_{dk}}+\upsilon_{I_{dk}}-1,\gamma_{F_{dk}}+\psi_{F_{dk}}+\upsilon_{F_{dk}}-1\rangle\right)$$
$$=\left(\langle\lambda_{A_{dk}},\lambda_{I_{dk}},\lambda_{F_{dk}}\rangle\right)$$

2.2) If $\gamma_{A_{dk}}+\psi_{A_{dk}}+\upsilon_{A_{dk}}>1$

$\gamma_{I_{dk}}+\psi_{I_{dk}}+\upsilon_{I_{dk}}-1>0$

$\gamma_{F_{dk}} + \psi_{F_{dk}} + \upsilon_{F_{dk}} - 1 < 0$ then,

$$(\langle \nu_{A_{dk}}, \nu_{I_{dk}}, \nu_{F_{dk}} \rangle) = \begin{pmatrix} \min(1, \gamma_{A_{dk}} + \psi_{A_{dk}} + \upsilon_{A_{dk}}), \max(0, \gamma_{I_{dk}} + \psi_{I_{dk}} + \upsilon_{I_{dk}} - 1 - 1), \\ \max(0, \gamma_{F_{dk}} + \psi_{F_{dk}} + \upsilon_{F_{dk}} - 1 - 1) \end{pmatrix}$$
$$= (\langle \gamma_{A_{dk}} + \psi_{A_{dk}} + \upsilon_{A_{dk}}, \gamma_{I_{dk}} + \psi_{I_{dk}} + \upsilon_{I_{dk}} - 1, 0 \rangle)$$
$$= (\langle \lambda_{A_{dk}}, \lambda_{I_{dk}}, \lambda_{F_{dk}} \rangle)$$

Therefore, in all cases

$$(\langle \nu_{A_{dk}}, \nu_{I_{dk}}, \nu_{F_{dk}} \rangle) = (\langle \lambda_{A_{dk}}, \lambda_{I_{dk}}, \lambda_{F_{dk}} \rangle)$$

$$(\Gamma \wedge_L^* \Psi) \wedge_L^* \Upsilon = \Gamma \wedge_L^* (\Psi \wedge_L^* \Upsilon)$$

Proposition 3.5.

Let Γ, Ψ, Υ be three NFMs. Then $\Gamma \vee_L^* (\Psi \vee \Upsilon) = (\Gamma \vee_L^* \Psi) \vee (\Gamma \vee_L^* \Upsilon)$

Proof:

Let $(\Psi \vee \Upsilon) = (\langle \varkappa_{A_{dk}}, \varkappa_{I_{dk}}, \varkappa_{F_{dk}} \rangle)$; $(\Gamma \vee_L^* \Psi) = (\langle \sigma_{A_{dk}}, \sigma_{I_{dk}}, \sigma_{F_{dk}} \rangle)$

$$(\Gamma \vee_L^* \Upsilon) = (\langle \tau_{A_{dk}}, \tau_{I_{dk}}, \tau_{F_{dk}} \rangle)$$
$$\Gamma \vee_L^* (\Psi \vee \Upsilon) = (\langle \omega_{A_{dk}}, \omega_{I_{dk}}, \omega_{F_{dk}} \rangle)$$
$$(\Gamma \vee_L^* \Psi) \vee (\Gamma \vee_L^* \Upsilon) = (\langle \lambda_{A_{dk}}, \lambda_{I_{dk}}, \lambda_{F_{dk}} \rangle)$$

Now, $(\langle \varkappa_{A_{dk}}, \varkappa_{I_{dk}}, \varkappa_{F_{dk}} \rangle) = (\langle (\max(\psi_{A_{dk}}, \upsilon_{A_{dk}}), \min(\psi_{I_{dk}}, \upsilon_{I_{dk}}), \min(\psi_{F_{dk}}, \upsilon_{F_{dk}}) \rangle)$

$$(\langle \sigma_{A_{dk}}, \sigma_{I_{dk}}, \sigma_{F_{dk}} \rangle) = (\langle (\max(0, \gamma_{A_{dk}} + \psi_{A_{dk}} - 1), \min(1, \gamma_{I_{dk}} + \psi_{I_{dk}}), \min(1, \gamma_{F_{dk}} + \psi_{F_{dk}}) \rangle)$$

$$(\langle \tau_{A_{dk}}, \tau_{I_{dk}}, \tau_{F_{dk}} \rangle) = (\langle (\max(0, \gamma_{A_{dk}} + \upsilon_{A_{dk}} - 1), \min(1, \gamma_{I_{dk}} + \upsilon_{I_{dk}}), \min(1, \gamma_{F_{dk}} + \upsilon_{F_{dk}}) \rangle)$$

$$(\langle \omega_{A_{dk}}, \omega_{I_{dk}}, \omega_{F_{dk}} \rangle) = (\langle \max(0, \gamma_{A_{dk}} + \varkappa_{A_{dk}} - 1), \min(1, \gamma_{I_{dk}} + \varkappa_{I_{dk}}), \min(1, \gamma_{F_{dk}} + \varkappa_{F_{dk}}) \rangle) \quad (3.11)$$

$$(< \lambda_{A_{dk}}, \lambda_{I_{dk}}, \lambda_{F_{dk}} >) = (\max(\sigma_{A_{dk}}, \tau_{A_{dk}}), \min(\sigma_{I_{dk}}, \tau_{I_{dk}}), \min(\sigma_{F_{dk}}, \tau_{F_{dk}})) \quad (3.12)$$

We have to prove (1) =(2)

Case 1)

Suppose

$$(\langle \psi_{A_{dk}}, \psi_{I_{dk}}, \psi_{F_{dk}} \rangle) \leq (\langle \upsilon_{A_{dk}}, \upsilon_{I_{dk}}, \upsilon_{F_{dk}} \rangle) \quad (3.13)$$

Then, $(\langle \varkappa_{A_{dk}}, \varkappa_{I_{dk}}, \varkappa_{F_{dk}} \rangle) = (\langle \upsilon_{A_{dk}}, \upsilon_{I_{dk}}, \upsilon_{F_{dk}} \rangle)$

$$(\langle \omega_{A_{dk}}, \omega_{I_{dk}}, \omega_{F_{dk}} \rangle) = (\langle \max(0, \gamma_{A_{dk}} + \varkappa_{A_{dk}} - 1), \min(1, \gamma_{I_{dk}} + \varkappa_{I_{dk}}), \min(1, \gamma_{F_{dk}} + \varkappa_{F_{dk}}) \rangle)$$

Either $\gamma_{A_{dk}} + \varkappa_{A_{dk}} - 1 \geq 0$ or $\gamma_{A_{dk}} + \varkappa_{A_{dk}} < 1$

$\gamma_{I_{dk}} + \varkappa_{I_{dk}} \geq 1$ and $\gamma_{F_{dk}} + \varkappa_{F_{dk}} \geq 1$

$$\left(\langle\omega_{A_{dk}},\omega_{I_{dk}},\omega_{F_{dk}}\rangle\right)=\left(\gamma_{A_{dk}}+\varkappa_{A_{dk}}-1,1,1\right) \quad (3.14)$$

Now, $\left(\langle\lambda_{A_{dk}},\lambda_{I_{dk}},\lambda_{F_{dk}}\rangle\right) = \left(\max\left(\sigma_{A_{dk}},\tau_{A_{dk}}\right),\min\left(\sigma_{I_{dk}},\tau_{A_{dk}}\right),\min\left(\sigma_{A_{dk}},\tau_{A_{dk}}\right)\right)$

Here,

$$\left(\langle\sigma_{A_{dk}},\sigma_{I_{dk}},\sigma_{F_{dk}}\rangle\right) = \left(\langle\max\left(0,\gamma_{A_{dk}}+\psi_{A_{dk}}-1\right),\min\left(1,\gamma_{I_{dk}}+\psi_{I_{dk}}\right),\min\left(1,\gamma_{F_{dk}}+\psi_{F_{dk}}\right)\rangle\right)$$

If $\gamma_{A_{dk}}+\psi_{A_{dk}}-1\geq 0$, $\gamma_{I_{dk}}+\psi_{I_{dk}}\geq 1$ and $\gamma_{F_{dk}}+\psi_{F_{dk}}\geq 1$ then

$$\left(\langle\sigma_{A_{dk}},\sigma_{I_{dk}},\sigma_{F_{dk}}\rangle\right)=\left(\gamma_{A_{dk}}+\psi_{A_{dk}}-1,1,1\right) \quad (3.15)$$

If $\gamma_{A_{dk}}+\upsilon_{A_{dk}}-1>0$, $\gamma_{I_{dk}}+\upsilon_{I_{dk}}\geq 1$ and $\gamma_{F_{dk}}+\upsilon_{F_{dk}}\geq 1$ then

$$\left(\langle\tau_{A_{dk}},\tau_{I_{dk}},\tau_{F_{dk}}\rangle\right) = \left(\langle\max\left(0,\gamma_{A_{dk}}+\upsilon_{A_{dk}}-1\right),\min\left(1,\gamma_{I_{dk}}+\upsilon_{I_{dk}}\right),\min\left(1,\gamma_{F_{dk}}+\upsilon_{F_{dk}}\right)\rangle\right) = \left(\gamma_{A_{dk}}+\upsilon_{A_{dk}}-1,1,1\right)$$

$$(\Gamma \vee_L^* \Psi)\vee(\Gamma \vee_L^* \Upsilon) = \left(\langle\max\left(\left(\gamma_{A_{dk}}+\psi_{A_{dk}}-1\right),\gamma_{A_{dk}}+\upsilon_{A_{dk}}-1\right)\right),\min(1,1),\min(1,1)\rangle\right)$$

By eq (3.13)

$$\left(\langle\lambda_{A_{dk}},\lambda_{I_{dk}},\lambda_{F_{dk}}\rangle\right)=\left(\gamma_{A_{dk}}+\upsilon_{A_{dk}}-1,1,1\right) \quad (3.16)$$

$$=\left(\gamma_{A_{dk}}+\varkappa_{A_{dk}}-1,1,1\right)$$

Therefore (3.13) = (3.16)

Similarly, we can prove for $\left(\langle\psi_{A_{dk}},\psi_{I_{dk}},\psi_{F_{dk}}\rangle\right)\geq\left(\langle\upsilon_{A_{dk}},\upsilon_{I_{dk}},\upsilon_{F_{dk}}\rangle\right)$

$$\Rightarrow \Gamma \vee_L^* (\Psi \vee \Upsilon) = (\Gamma \vee_L^* \Psi)\vee(\Gamma \vee_L^* \Upsilon)$$

4. Conclusion

Here we overcome some properties of Lukasiewicz over NFMs. Also, we worked commutative, reflexive, irreflexive over $\vee_L^*$ and $\wedge_L^*$. We developed $\vee_L^*$ and $\wedge_L^*$ in NFMs. In day today life we are facing any problem in indeterministic form. These and all solved using neutrosophic fuzzy sets and matrices. Here we solved some property using max,min,min and min,max,max in NFMs. And also, we travelled with Lukasiewicz concepts in NFMs.

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