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FUZZY ALGEBRA OVER FERMATEAN FUZZY MATRICES

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Abstract

The concept of semi ring of Fermatean fuzzy matrices is explored in this research (FFMs). The FFMs are shown to constitute a Fermatean fuzzy algebra. Using the comparability to investigate some of the properties of FFMs. Then the trace and transpose of Fermatean fuzzy matrices are defined accordingly and some properties are studied.

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1 Introduction

Atanassov [1] extended the idea of an intuitionistic fuzzy set (IFS), which consists of a set's element x's membership, non-membership, and hesitation degrees. IFMs shall be defined by Im.et.al [8] as a natural generalization of fuzzy matrices, and the determinant of square IFMs was explored. The investigation of certain IFM operations were carried out by Khan and M. Pal [7]. Zadeh [9] established the idea of FS in 1965, deals with imprecision and ambiguity in practical circumstances. The idea of decision-making problems containing uncertainty was first proposed by Belmann and Zadeh (9) in 1970.Only the complementary degree of membership function and degree of non-membership function are accepted by the FS. The sum of the membership function and the non-membership function may, on occasion, be more than one. As a result, orthopair fuzzy sets (FSs) are created, in which an element's membership grades are expressed as pairs of values in the unit interval $\eta(x)$, $\zeta(x)$, where one value denotes support for membership in the fuzzy set and the other denotes support against membership. For instance, Atanassov's second class of intuitionistic fuzzy sets (IFSs) [3] and his conventional IFSs [1, 2].

A new orthopair of FS, known as a Pythagorean fuzzy set (PFS), was recently presented by Yager [4],[5], where the square sum of the membership value and non-membership value is equal to or less than one. In a short period of time, PFSs have captured the interest of numerous researchers, and a number of researchers have also suggested real-world uses in a Pythagorean fuzzy context. However, if the orthopair fuzzy set to <0.88, 0.55>, where 0.88 is the support for the membership of a particular parameter's criteria and 0.55 is the support against non-membership, it will not adhere to the IFS and PFS conditions. The support for membership and

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the support against membership degrees, however, add up to a number that is either equal to one or less. Additionally, Senapati T and Yager R R [6] very just introduced the Fermatean Fuzzy set in this circumstance (FFS). They also demonstrated that FFSs can handle larger levels of uncertainty than IFSs and PFSs.

Section 2 of this paper offers the introduction and some background information for this investigation. In section 3, we demonstrated that $\mathcal{F}_n$ is a Fermatean fuzzy algebra for scalar multiplication, component wise addition and component wise multiplication. Also proved that associative and distributive properties are satisfied under matrix multiplication in Fermatean fuzzy matrix. Some qualities are also proved using the notion of FFM comparability.

2. Preliminaries

Definition 2.1: A fuzzy matrix (FM) of order a x b is defined as R=$\left(\langle \eta_{r_{ij}} \rangle\right)$, where $\eta_{r_{ij}}$∈[0, 1] and $1 \le i \le a, 1 \le j \le b$.

Definition 2.2: An Intuitionistic Fuzzy Matrix (IFM) of order a x b is defined as R=$\left(\langle \eta_{r_{ij}}, \zeta_{r_{ij}} \rangle\right)$, is a matrix of nonnegative numbers $\eta_{r_{ij}}, \zeta_{r_{ij}}$ ∈ [0, 1], satisfying $0 \le \eta_{a_{ij}} + \zeta_{a_{ij}} \le 1$ for all $1 \le i \le a, 1 \le j \le b$

Definition 2.3: A Fermatean Fuzzy Matrix (FFM) of order a x b is defined as A=$\left(\langle \eta_{r_{ij}}, \zeta_{r_{ij}} \rangle\right)$is a matrix of nonnegative numbers $\eta_{r_{ij}}, \zeta_{r_{ij}}$ ∈ [0, 1], satisfying

$$0 \le \left(\left(\eta_{r_{ij}}\right)^3 + \left(\zeta_{r_{ij}}\right)^3\right) \le 1 \text{ for every } i, j.$$

Let $\mathcal{F}_{ab}$ be the collection of all Fermatean fuzzy matrices of order a x b

Definition 2.4: Let $R = \left(\langle \eta_{r_{ij}}, \zeta_{r_{ij}} \rangle\right)$ *and* $S = \left(\langle \eta_{s_{ij}}, \zeta_{b_{ij}} \rangle\right)$, then the matrix component-by-component addition and multiplication are provided by

$$R + S = \left(\langle max\left(\eta_{r_{ij}}, \eta_{s_{ij}}\right), min\left(\zeta_{r_{ij}}, \zeta_{s_{ij}}\right)\rangle\right). \in \mathcal{F}_{ab}$$

$$R \bullet S = \left(\langle min\left(\eta_{r_{ij}}, \eta_{s_{ij}}\right), max\left(\zeta_{r_{ij}}, \zeta_{s_{ij}}\right)\rangle\right)$$

Definition 2.5: Let $R = \left(\langle \eta_{r_{ij}}, \zeta_{r_{ij}} \rangle\right) \in \mathcal{F}_{ab}$, $S = \left(\langle \eta_{s_{ij}}, \zeta_{b_{ij}} \rangle\right) \in \mathcal{F}_{bc}$, then the matrix multiplication is defined as $RS = \left(\langle \max\left\{min\left(\eta_{r_{ik}}, \eta_{s_{kj}}\right)\right\}, \min\left\{max\left(\zeta_{r_{ik}}, \zeta_{s_{kj}}\right)\right\}\rangle\right)$ where $1 \le k \le c$, $1 \le i \le a, 1 \le j \le b$ whenever R and S are comparable matrices.

Definition 2.6: If every entry is $(\langle 0,1 \rangle)$, the matrix is the Zero Fermatean fuzzy matrix of order m x n.

Indicated as I_n, the identity matrix of order n x n is defined such that $(\langle 1,0 \rangle)$ if i = j and $(\langle 0,1 \rangle)$ if $i \ne j$

If all of the elements are $\langle 1,0 \rangle$, then the matrix is the universal matrix (J) of order m x n.

Definition 2.7: Let $R = \left(\langle \eta_{r_{ij}}, \zeta_{r_{ij}} \rangle\right) \in \mathcal{F}_{ab}$ and $\mu \in F$, then $\mu R = \left(\langle min\left(\mu, \eta_{r_{ij}}\right), max\left(1 - \mu, \zeta_{r_{ij}}\right)\rangle\right) \in \mathcal{F}_{ab}$. is the definition of the Fermatean fuzzy scalar multiplication

By definition, for the universal matrix J

$\mu J = (\langle min(\mu, 1), max(1-\mu, 0)\rangle) = (\langle \mu, 1-\mu\rangle)$

Under component wise multiplication,

$\mu J \bullet R = \left(\langle min\left(\mu, \eta_{r_{ij}}\right), max\left(1-\mu, \zeta_{r_{ij}}\right)\rangle\right) = \mu R$

Definition 2.8: Let $R = \left(\langle \eta_{r_{ij}}, \zeta_{r_{ij}}\rangle\right)$, $S = \left(\langle \eta_{s_{ij}}, \zeta_{b_{ij}}\rangle\right) \in \mathcal{F}_{ab}$. R and S are said to be comparable (R ≤ S) if $\eta_{r_{ij}} \le \eta_{s_{ij}}$ and $\zeta_{r_{ij}} \ge \zeta_{s_{ij}}$ for all i, j.

For all R $\in \mathcal{F}_{ab}$, R+J = J and R • 0 = 0 shows that the existence of the universal bound of R.

Definition 2.9: If every row and column include precisely one $\langle 1,0\rangle$ and all other entries are $\langle 0,1\rangle$, then the Fermatean fuzzy permutation matrix is a square Fermatean fuzzy matrix. Assume that Pb is the set of all a x b such matrices in F_b. $RR^T=R^TR=I_b$, where R^T is the transpose of R, if R $\in P_b$

Section 3:

In this section we prove that $\mathcal{F}_b$ is a Fermatean fuzzy algebra under the component wise addition, component wise multiplication.

Theorem 3.1: The set $\mathcal{F}_{ab}$ is a Fermatean fuzzy algebra when addition and multiplication are performed (+,•) component by component.

Proof: R + 0 = R and R •J = R for all R $\in \mathcal{F}_{ab}$ are evidently equal.

The zero matrix is hence the additive identity and the multiplicative identity is the universal matrix J shows that the existence of the identity element with respect to the operation + and •

For $R = \left(\langle \eta_{r_{ij}}, \zeta_{r_{ij}}\rangle\right)$, $S = \left(\langle \eta_{s_{ij}}, \zeta_{s_{ij}}\rangle\right)$ $and\ T = \left(\langle \eta_{t_{ij}}, \zeta_{t_{ij}}\rangle\right) \in \mathcal{F}_{ab}$

$$R+(S+T) = \left(\langle \eta_{r_{ij}}, \zeta_{r_{ij}}\rangle\right) + \left(\langle max\left(\eta_{s_{ij}}, \eta_{t_{ij}}\right), min\left(\zeta_{s_{ij}}, \zeta_{t_{ij}}\right)\rangle\right)$$

$$= \left(\langle max\left(\eta_{r_{ij}}, \eta_{s_{ij}}, \eta_{t_{ij}}\right), min\left(\zeta_{r_{ij}}, \zeta_{s_{ij}}, \zeta_{t_{ij}}\right)\rangle\right) \quad (3.1.1)$$

$$(R+S)+T = \left(\langle max\left(\eta_{r_{ij}}, \eta_{s_{ij}}\right), min\left(\zeta_{r_{ij}}, \zeta_{s_{ij}}\right)\rangle\right) + \left(\langle \eta_{t_{ij}}, \zeta_{t_{ij}}\rangle\right)$$

$$= \left(\langle max\left(\eta_{r_{ij}}, \eta_{s_{ij}}, \eta_{t_{ij}}\right), min\left(\zeta_{r_{ij}}, \zeta_{s_{ij}}, \zeta_{t_{ij}}\right)\rangle\right) \quad (3.1.2)$$

From (3.1.1) and (3.1.2) we have R+(S+T) =(R+S) +T.

To prove R• $(S \bullet T) = (R \bullet S) \bullet T$

$$R \bullet (S \bullet T) = \left(\langle \eta_{r_{ij}}, \zeta_{r_{ij}}\rangle\right) \bullet \left(\langle min\left(\eta_{s_{ij}}, \eta_{t_{ij}}\right), max\left(\zeta_{s_{ij}}, \zeta_{t_{ij}}\right)\rangle\right)$$

$$= \left(\langle min\left(\eta_{r_{ij}}, \eta_{s_{ij}}, \eta_{t_{ij}}\right), max\left(\zeta_{r_{ij}}, \zeta_{s_{ij}}, \zeta_{t_{ij}}\right)\rangle\right) \quad (3.1.3)$$

$$(R \bullet S) \bullet T = \left(\langle min\left(\eta_{r_{ij}}, \eta_{s_{ij}}\right), max\left(\zeta_{r_{ij}}, \zeta_{s_{ij}}\right)\rangle\right) \bullet \left(\langle \eta_{t_{ij}}, \zeta_{t_{ij}}\rangle\right)$$

$$= \left(\langle min\left(\eta_{r_{ij}}, \eta_{s_{ij}}, \eta_{t_{ij}}\right), max\left(\zeta_{r_{ij}}, \zeta_{s_{ij}}, \zeta_{t_{ij}}\right)\rangle\right) \quad (3.1.4)$$

From (3.1.3) and (3.1.4) we have R• $(S \bullet T) = (R \bullet S) \bullet T$ which proves that associativity is satisfied under (+,•).

Also R + (R• S) $= \left(\langle \eta_{r_{ij}}, \zeta_{r_{ij}}\rangle\right) + \left(\langle min\left(\eta_{r_{ij}}, \eta_{s_{ij}}\right), max\left(\zeta_{r_{ij}}, \zeta_{s_{ij}}\right)\rangle\right)$

$$= \left(\langle max\left(\eta_{r_{ij}}, min\left(\eta_{r_{ij}}, \eta_{s_{ij}}\right)\right), min\left(\zeta_{r_{ij}}, max\left(\zeta_{r_{ij}}, \zeta_{s_{ij}}\right)\right)\rangle\right)$$

$=\left(\langle \eta_{r_{ij}}, \zeta_{r_{ij}}\rangle\right)$ =R

R• $(R+S)=\left(\langle \eta_{r_{ij}}, \zeta_{r_{ij}}\rangle\right) \bullet \left(\langle max\left(\eta_{r_{ij}}, \eta_{s_{ij}}\right), min\left(\zeta_{r_{ij}}, \zeta_{s_{ij}}\right)\rangle\right)$

$=\left(\langle min\left(\eta_{r_{ij}}, max\left(\eta_{r_{ij}}, \eta_{s_{ij}}\right)\right), max\left(\zeta_{r_{ij}}, min\left(\zeta_{r_{ij}}, \zeta_{s_{ij}}\right)\right)\rangle\right)$

$=\left(\langle \eta_{r_{ij}}, \zeta_{r_{ij}}\rangle\right)$ =R

Hence $\mathcal{F}_{ab}$ satisfies the condition for absorption.

To Prove the Distributive Property:

Case I: Assume R ≤ S or T

R• (S+T) $=\left(\langle min\left(\eta_{r_{ij}}, max\left(\eta_{s_{ij}}, \eta_{t_{ij}}\right)\right), max\left(\zeta_{r_{ij}}, min\left(\zeta_{s_{ij}}, \zeta_{t_{ij}}\right)\right)\rangle\right) = \left(\langle \eta_{r_{ij}}, \zeta_{r_{ij}}\rangle\right)$ = R

$=\left(\langle \eta_{r_{ij}}, \zeta_{r_{ij}}\rangle\right)$ = R (3.1.5)

(R •S)+(R•T) $=\left(\langle min\left(\eta_{r_{ij}}, \eta_{s_{ij}}\right), max\left(\zeta_{r_{ij}}, \zeta_{s_{ij}}\right)\rangle\right) + \left(\langle min\left(\eta_{r_{ij}}, \eta_{t_{ij}}\right), max\left(\zeta_{r_{ij}}, \zeta_{t_{ij}}\right)\rangle\right)$

$= \left(\langle max\left(min\left(\eta_{r_{ij}}, \eta_{s_{ij}}\right), min\left(\eta_{r_{ij}}, \eta_{t_{ij}}\right)\right), min\left(max\left(\zeta_{r_{ij}}, \zeta_{s_{ij}}\right), max\left(\zeta_{r_{ij}}, \zeta_{t_{ij}}\right)\right)\rangle\right)$

$=\left(\langle \eta_{r_{ij}}, \zeta_{r_{ij}}\rangle\right)$ = R (3.1.6)

From (3.1.5) and (3.1.6) R• (S+T) = (R •S) + (R•T)

Case II: Assume R ≥ S and T

Subcase 1: R ≥ S ≥ T

Using (5) and (6) in Case I, R• (S+T) = S = (R •S) +(R•T)

Subcase 2: R ≥ T ≥ S

Using (5) and (6) in Case I, R• (S+T) = T = (R •S) +(R•T)

Hence from Case I & Case II, $\mathcal{F}_{ab}$ holds the distributive property.

In a similar manner we can prove that (R +S) •T = (R •T) +(S•T)

Therefore we proved $\mathcal{F}_{ab}$ is Fermatean Fuzzy Algebra under (+,•).

Remark: Let R, S, T∈ $\mathcal{F}_{ab}$ .From the above results we have R+S= S+R, R• S=S •R

Also associative property, commutative property are satisfied in $\mathcal{F}_{ab}$.

For any $\mu \in F$, $\mu(S+T) = \mu J \bullet (S+T)$

$= \mu J \bullet S + \mu J \bullet T$ (using theorem 3.1)

$= \mu S + \mu T$

Let $\mu_1, \mu_2 \in$ F , $(\mu_1 + \mu_2)T = (\mu_1 + \mu_2)J \bullet T$

$=(\mu_1 J + \mu_2 J) \bullet T$

$= \mu_1 J \bullet T + \mu_2 J \bullet T = \mu_1 T + \mu_2 T$

Hence $\mathcal{F}_{ab}$ is a Fermatean vector space over F.

Theorem 3.2: R, S, T are any three FFMs of order a x b, b x c, c x d respectively then

(i) (RS)T =R(ST)

(ii) R(S+T) =RS +RT

Proof: Let $R = \left(\langle \eta_{r_{ij}}, \zeta_{r_{ij}}\rangle\right)$, $S = \left(\langle \eta_{s_{jk}}, \zeta_{s_{jk}}\rangle\right)$ *and* $T = \left(\langle \eta_{t_{kl}}, \zeta_{t_{kl}}\rangle\right)$

where i ranges from 1 to a, j ranges from 1 to b, k ranges from 1 to c and l varies from 1 to d respectively.

To Prove (i):The product's (i,k) th component is represented by the expression

$RS = \left(\langle \sum_{j=1}^{b} \eta_{r_{ij}}\eta_{s_{jk}}, \prod_{j=1}^{b} \zeta_{r_{ij}} + \zeta_{s_{jk}} \rangle\right)$

The sum of the products of the corresponding components in the i^{th} row of RS, first column of T, with k common, makes up the $(i,1)^{th}$ element in the product (RS)T.

The $(i,1)^{th}$ element of $(RS)T = \langle \sum_{k=1}^{c} \left(\sum_{j=1}^{b} \eta_{r_{ij}}\eta_{s_{jk}}\right)\eta_{t_{kl}}, \prod_{k=1}^{c}\left(\prod_{j=1}^{b}\left(\zeta_{r_{ij}}+\zeta_{s_{jk}}\right)+\zeta_{t_{kl}}\right)\rangle$

$$=\langle \sum_{k=1}^{c}\sum_{j=1}^{b} \eta_{r_{ij}}\eta_{s_{jk}}\eta_{t_{kl}}, \prod_{k=1}^{c}\prod_{j=1}^{b}\left(\zeta_{r_{ij}}+\zeta_{s_{jk}}+\zeta_{t_{kl}}\right)\rangle \quad (3.2.1)$$

The $(j,1)^{th}$ element of the product $ST = \langle \sum_{k=1}^{c} \eta_{s_{jk}}\eta_{t_{kl}}, \prod_{k=1}^{c}\left(\zeta_{s_{jk}}+\zeta_{t_{kl}}\right)\rangle$

The sum of the products of the corresponding components in the i^{th} row of R and the first column of ST makes up the $(i,1)^{th}$ element of the product R(ST).

The $(i,l)^{th}$ element of $R(ST) = \langle \sum_{j=1}^{b} \eta_{r_{ij}}\left(\sum_{k=1}^{c} \eta_{s_{jk}}\eta_{t_{kl}}\right), \prod_{j=1}^{b}\left(\zeta_{r_{ij}} + \prod_{k=1}^{c}\left(\zeta_{s_{jk}}+\zeta_{t_{kl}}\right)\right)\rangle$

$$=\langle \sum_{k=1}^{c}\sum_{j=1}^{b} \eta_{r_{ij}}\eta_{s_{jk}}\eta_{t_{kl}}, \prod_{k=1}^{c}\prod_{j=1}^{b}\left(\zeta_{r_{ij}}+\zeta_{s_{jk}}+\zeta_{t_{kl}}\right)\rangle \quad (3.2.2)$$

Combining (3.2.1) and (3.2.2) R (ST) = (RS) T

To Prove (ii): Consider the (j,k) th element of $S + T = \left(\langle max\left(\eta_{s_{jk}}, \eta_{t_{jk}}\right), min\left(\zeta_{s_{jk}}, \zeta_{t_{jk}}\right)\rangle\right)$

$$=\left(\langle \eta_{s_{jk}} + \eta_{t_{jk}}, \ \zeta_{s_{jk}}\zeta_{t_{jk}} \rangle\right),$$

Take into account that the k^{th} element in the product of R and S+T is equal to the sum of the products of the equivalent components in the i^{th} row of R and the kth element of S+T.

$$R(S+T) = \left(\langle \sum_{j=1}^{b} \eta_{r_{ij}}\left(\eta_{s_{jk}} + \eta_{t_{jk}}\right), \prod_{j=1}^{b} \zeta_{r_{ij}}\left(\zeta_{s_{jk}}\zeta_{t_{jk}}\right)\rangle\right) \text{------------- } (3.2.3)$$

$$RS + RT = \left(\langle \sum_{j=1}^{b} \eta_{r_{ij}}\eta_{s_{jk}}, \prod_{j=1}^{b}\left(\zeta_{r_{ij}} + \zeta_{s_{jk}}\right)\rangle\right) + \left(\langle \sum_{j=1}^{b} \eta_{r_{jk}}\eta_{t_{jk}}, \prod_{j=1}^{b}\left(\zeta_{r_{jk}} + \zeta_{t_{jk}}\right)\rangle\right)$$

$$=\left(\langle \sum_{j=1}^{b}\left(\eta_{r_{ij}}\eta_{s_{jk}} + \eta_{r_{jk}}\eta_{t_{jk}}\right), \prod_{j=1}^{b}\left(\zeta_{r_{ij}} + \zeta_{s_{jk}}\right)\prod_{j=1}^{b}\left(\zeta_{r_{jk}} + \zeta_{t_{jk}}\right)\rangle\right)$$

$$= \left(\langle \sum_{j=1}^{b} \eta_{r_{ij}}\left(\eta_{s_{jk}} + \eta_{t_{jk}}\right), \prod_{j=1}^{b}\left(\zeta_{r_{ij}}\left(\zeta_{s_{jk}}\zeta_{t_{jk}}\right)\right)\rangle\right) \text{-------------------------- } (3.2.4)$$

Combining (3.2.3) and (3.2.4) we have proved (ii)

Theorem 3.3: Let R, S∈ $\mathcal{F}_{ab}$ then $R \leq S \text{ if and only if } R + S = S$

Proof: If $R \leq S$ then R+S= $\left(\langle max\left(\eta_{r_{ij}}, \eta_{s_{ij}}\right), min\left(\zeta_{r_{ij}}, \zeta_{s_{ij}}\right)\rangle\right) = \left(\langle \eta_{s_{ij}}, \zeta_{s_{ij}} \rangle\right)$ = S

Conversely assume $R + S = S$ then $\eta_{r_{ij}} \leq \eta_{s_{ij}}$ and $\zeta_{r_{ij}} \geq \zeta_{s_{ij}}$ Which implies $R \leq S$.

Theorem 3.4: Let R, S∈ $\mathcal{F}_{ab}$ if $R \leq S$ then (i) $RT \leq ST$ for any T∈ $\mathcal{F}_{bc}$

(ii) $AR \leq AS$ for any A∈ $\mathcal{F}_{ca}$

Proof: If $R \leq S$ then $\eta_{r_{ik}} \leq \eta_{s_{ik}}$ and $\zeta_{r_{ik}} \geq \zeta_{s_{ik}}$ for i =1 to a, k =1 to c.

$\eta_{r_{ik}}\eta_{t_{kj}} \leq \eta_{s_{ik}}\eta_{t_{kj}}$ and $\zeta_{r_{ik}}\zeta_{t_{kj}} \geq \zeta_{s_{ik}}\zeta_{t_{kj}}$ for j =1 to b (By Fuzzy Multiplication)

Using Fuzzy addition $\sum_{k=1}^{c} \eta_{r_{ik}}\eta_{t_{kj}} \leq \sum_{k=1}^{c} \eta_{s_{ik}}\eta_{t_{kj}}$ and $\sum_{k=1}^{c} \zeta_{r_{ik}}\zeta_{t_{kj}} \geq \sum_{k=1}^{c} \zeta_{s_{ik}}\zeta_{t_{kj}}$

We get $RT \leq ST$

$\eta_{a_{ik}}\eta_{r_{kj}} \leq \eta_{a_{ik}}\eta_{s_{kj}}$ and $\zeta_{a_{ik}}\zeta_{r_{kj}} \geq \zeta_{a_{ik}}\zeta_{r_{kj}}$ for j =1 to b (By Fuzzy Multiplication)

Using Fuzzy addition $\sum_{k=1}^{c}\eta_{a_{ik}}\eta_{r_{kj}} \leq \sum_{k=1}^{c}\eta_{a_{ik}}\eta_{s_{kj}}$ and $\sum_{k=1}^{c}\zeta_{a_{ik}}\zeta_{r_{kj}} \geq \sum_{k=1}^{c}\zeta_{a_{ik}}\zeta_{r_{kj}}$
We get $AR \leq AS$

Section 4 –Trace, Transpose and its properties of Fermatean Fuzzy Matrices

Definition 4.1: Let $R = \left(\langle\eta_{r_{ij}}, \zeta_{r_{ij}}\rangle\right) \in \mathcal{F}_{ab}$ be FFM. Then the transpose of a Fermatean Fuzzy matrix of order a x b is defined as by transposing rows into columns and columns into rows is known as the transpose of R and is represented by the symbol R^t, which has the order b x a.

$R^t = \left(\langle\eta_{r_{ji}}, \zeta_{r_{ji}}\rangle\right)$

Definition 4.2: Let R and S be Fermatean Fuzzy square Matrices of same order then the matrix multiplication of R and S is defined as $RS = \sum_i \left(\langle \text{maxmin}\left(\eta_{r_{ij}}, \eta_{s_{ij}}\right), minmax(\zeta_{r_{ij}}, \zeta_{s_{ij}})\rangle\right)$

Properties of the Transpose of Fermatean Fuzzy Matrix

Theorem 4.3: If R^t and S^t are the transposes of the Fermatean Fuzzy Matrices R and S respectively, then the subsequent assertions are true.

(i) $(R^t)^t=R$
(ii) $(R+S)^t = R^t + S^t$,R and S of same order.
(iii) $(RS)^t = S^t R^t$, R and S are Fermatean Fuzzy square Matrices of same order.

Proof: To prove(i)

Let R be a Fermatean Fuzzy Matrix $R = \left(\langle\eta_{r_{ij}}, \zeta_{r_{ij}}\rangle\right) \in \mathcal{F}_{ab}$

Then the transpose of R is $R^t = \left(\langle\eta_{r_{ji}}, \zeta_{r_{ji}}\rangle\right) \in \mathcal{F}_{ba}$

The transpose of R^t is $(R^t)^t = \left(\langle\eta_{r_{ij}}, \zeta_{r_{ij}}\rangle\right) = R \in \mathcal{F}_{ab}$ which is of order a x b.

To Prove (ii) : Let R and S be any two Fermatean Fuzzy Matrices of same order.

Let $R = \left(\langle\eta_{r_{ij}}, \zeta_{r_{ij}}\rangle\right)$, $S = \left(\langle\eta_{s_{ij}}, \zeta_{s_{ij}}\rangle\right) \in \mathcal{F}_{ab}$

$R+S= \left(\langle max\left(\eta_{r_{ij}}, \eta_{s_{ij}}\right), min\left(\zeta_{r_{ij}}, \zeta_{s_{ij}}\right)\rangle\right) = \left(\langle\eta_{p_{ij}}, \zeta_{p_{ij}}\rangle\right)=P$

$R^t+S^t= \left(\langle max\left(\eta_{r_{ji}}, \eta_{s_{ji}}\right), min\left(\zeta_{r_{ji}}, \zeta_{s_{ji}}\right)\rangle\right) = \left(\langle\eta_{p_{ji}}, \zeta_{p_{ji}}\rangle\right)=P^t = (R+S)^t$

$(R+S)^t = R^t+S^t$

To prove (iii): Let R and S be any two Fermatean Fuzzy Square Matrices of same order.

$$RS = \sum_i \left(\langle \text{maxmin}\left(\eta_{r_{ij}}, \eta_{s_{ij}}\right), minmax(\zeta_{r_{ij}}, \zeta_{s_{ij}})\rangle\right)$$

$(RS)^t = \sum_j \left(\langle \text{maxmin}\left(\eta_{r_{ji}}, \eta_{s_{ji}}\right), minmax(\zeta_{r_{ji}}, \zeta_{s_{ji}})\rangle\right)$ ------------- (4.3.1)

$S^t = \left(\langle\eta_{s_{ji}}, \zeta_{s_{ji}}\rangle\right)$ $R^t = \left(\langle\eta_{r_{ji}}, \zeta_{r_{ji}}\rangle\right)$ be Fermatean fuzzy matrices of same order.

$S^t R^t = \sum_j \left(\langle \text{maxmin}\left(\eta_{r_{ji}}, \eta_{s_{ji}}\right), minmax(\zeta_{r_{ji}}, \zeta_{s_{ji}})\rangle\right)$ ---------------(4.3.2)

From (4.3.1) and (4.3.2) $(RS)^t = S^t R^t$

Definition 4.4: Let R be a Fermatean Fuzzy Square Matrix of order m. The definition of tr(R), which stands for the matrix R's trace, is $tr(R) = \left(\langle max(\eta_{r_{ii}}), min(\zeta_{r_{ii}})\rangle\right)$where $\eta_{r_{ii}}$ is the membership value and $\zeta_{r_{ii}}$ is the non-membership lies on the principal diagonal.

Theorem 4.5 : Let R and S be any two Fermatean Fuzzy Square Matrices of same order m and κ be any scalar such that 0< κ <1.Then

(i) tr (R+S)= tr R + tr S.

(ii) tr(κR) = κtr(R).

(iii) tr (RS) =tr(SR)

(iv) tr R=tr R^t ,where R^t is the transpose of R.

Proof: Let $R = \left(\langle \eta_{r_{ij}}, \zeta_{r_{ij}} \rangle\right)$, $S = \left(\langle \eta_{s_{ij}}, \zeta_{s_{ij}} \rangle\right) \in \mathcal{F}_{mm}$

To prove (i) : tr(R+S)= tr R + tr S.

R+S= $\left(\langle max\left(\eta_{r_{ij}}, \eta_{s_{ij}}\right), min\left(\zeta_{r_{ij}}, \zeta_{s_{ij}}\right)\rangle\right) = \left(\langle \eta_{p_{ij}}, \zeta_{p_{ij}} \rangle\right)$ = P

$tr\ (R + S) = \left(\langle max(\eta_{p_{ii}}), min(\zeta_{s_{ii}})\rangle\right)$-------------------------(4.5.1)

$tr(R) = \left(\langle \max\left(\eta_{r_{ii}}\right), \min\left(\zeta_{r_{ii}}\right)\rangle\right)$

$tr(S) = \left(\langle max(\eta_{s_{ii}}), min(\zeta_{s_{ii}})\rangle\right)$

$tr\ (R) + tr\ (S) = \left(\langle max\ \left(max\left(\eta_{r_{ii}}\right), \max\left(\eta_{s_{ii}}\right)\right), min\left(\min\left(\zeta_{r_{ii}}\right), min(\zeta_{s_{ii}})\right)\rangle\right)$

$= \left(\langle max\left(\eta_{r_{ii}}, \eta_{s_{ii}}\right), min\left(\zeta_{r_{ii}}, \zeta_{s_{ii}}\right)\rangle\right) = \left(\langle \eta_{p_{ii}}, \zeta_{p_{ii}} \rangle\right)$ = tr(P)---------(4.5.2)

From (4.5.1) and (4.5.2) tr(R+S)= tr R + tr S.

Example: $R = \begin{pmatrix} \langle .9, .6\rangle & \langle .5, .8\rangle \\ \langle .7, .7\rangle & \langle .8, .6\rangle \end{pmatrix}$ $S = \begin{pmatrix} \langle .5, .9\rangle & \langle .7, .8\rangle \\ \langle .5, .6\rangle & \langle .9, .2\rangle \end{pmatrix}$ $R + S = \begin{pmatrix} \langle .9, .6\rangle & \langle .7, .8\rangle \\ \langle .7, .6\rangle & \langle .9, .2\rangle \end{pmatrix}$

$tr(R + S) = (\langle .9, .2\rangle)$

$tr\ (R) = (\langle .9, .6\rangle)$ $tr\ (S) = (\langle .9, .2\rangle)$ tr (R) +tr (S) =$(\langle .9, .2\rangle)$

tr(R+S)= tr R + tr S.

To Prove (ii) :

Let $R = \left(\langle \eta_{r_{ij}}, \zeta_{r_{ij}} \rangle\right)$ κR =$\left(\langle \kappa\eta_{r_{ij}}, \kappa\zeta_{r_{ij}} \rangle\right)$, 0 < κ <1

tr (κR) =$\left(\langle \kappa\eta_{r_{ii}}, \kappa\zeta_{r_{ii}} \rangle\right)$ =$\left(\kappa\langle \eta_{r_{ii}}, \zeta_{r_{ii}} \rangle\right)$

= κtr (R)

To Prove (iii):

$$RS = \sum_i \left(\langle \text{maximin}\left(\eta_{r_{ij}}, \eta_{s_{ij}}\right), minmax(\zeta_{r_{ij}}, \zeta_{s_{ij}})\rangle\right)$$

$SR = \sum_i \left(\langle \text{maximin}\left(\eta_{s_{ij}}, \eta_{r_{ij}}\right), minmax(\zeta_{s_{ij}}, \zeta_{r_{ij}})\rangle\right)$ = RS

Therefore tr(RS)= tr(SR).

Example:

$R = \begin{pmatrix} \langle .5, .9\rangle & \langle .8, .4\rangle \\ \langle .6, .9\rangle & \langle .8, .3\rangle \end{pmatrix}$ $S = \begin{pmatrix} \langle .6, .8\rangle & \langle .9, .4\rangle \\ \langle .7, .6\rangle & \langle .9, .5\rangle \end{pmatrix}$ $R = \begin{pmatrix} \langle .5, .9\rangle & \langle .8, .4\rangle \\ \langle .6, .9\rangle & \langle .8, .3\rangle \end{pmatrix}$

$RS = \begin{pmatrix} \langle .7, .6\rangle & \langle .8, .5\rangle \\ \langle .7, .6\rangle & \langle .8, .5\rangle \end{pmatrix}$ $SR = \begin{pmatrix} \langle .6, .9\rangle & \langle .8, .4\rangle \\ \langle .6, .9\rangle & \langle .8, .5\rangle \end{pmatrix}$

$tr\ (RS) = (\langle .8, .5\rangle)$

$tr\ (SR) = (\langle .8, .5\rangle)$

To Prove (iv): Let $R = \left(\langle \eta_{r_{ij}}, \zeta_{r_{ij}} \rangle\right)$, $R^t = \left(\langle \eta_{r_{ji}}, \zeta_{r_{ji}} \rangle\right) \in \mathcal{F}_{mm}$

$tr(R) = \left(\langle \max\left(\eta_{r_{ii}}\right), \min\left(\zeta_{r_{ii}}\right)\rangle\right)$ -------------(4.5.3)

$tr(R^t) = (\langle \max(\eta_{r_{ii}}), \min(\zeta_{r_{ii}}) \rangle)$--------------(4.5.4)

From (4.5.3) and (4.5.4) $tr(R) = tr(R^t)$

Application:

There is still to examine more set attributes resulting from other defining set operations that could be thought of by combining the functions of η , ζ in different ways. Then the properties of trace and transpose of Fermatean fuzzy matrices are defined and proved. We may use these operators in future research in a variety of fields, including dynamic decision making and consensus, business and marketing management, design, engineering, and manufacturing, information technology and networking applications, human resources management, military applications, energy management, geographic information system applications, medical diagnosis and transportation problems etc.

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