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STRUCTURAL PROPERTIES OF MININORMED EUCLIDEAN SPACES

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Abstract. A mininorm on a vector function X and field as $\mathbb{K} = \mathbb{R}$, such that $\mathbb{C}$ is a function ω from X to $\mathbb{R}$ satisfying the properties of a norm $\|\ \|$ with the property $\|\alpha x\| = |\alpha|\ \|x\|$, $\alpha \in \mathbb{K}$, $x \in X$ replaced by the following property

$$\|\alpha x\| = \|x\|\ \forall\ x \in X, \alpha \notin 0.$$

There are several mininorms on the Euclidean spaces $\mathbb{R}^k$. One such mininorm is the Hamming weight function. In this paper, we discuss certain basic properties of Euclidean spaces with the Hamming mininorm and also some structural properties of these spaces.

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1. Introduction

In coding theory, the Hamming weight $\omega_H(x)$, of a code word x is defined to be the number of non-zero coordinates of x.

$\omega_H(x)$ satisfies the following conditions:

(i) $\omega_H(x) \geq 0$ for all code words x and $\omega_H(x) = 0$ if and only if $x = 0$.

(ii) $\omega_H(x + y) \leq \omega_H(x) + \omega_H(y)$ for all code words x and y

Also, ω_H satisfies an additional property.

$\omega_H(\alpha x) = \omega_H(x)$ for all $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}, \alpha \notin 0$.

Thus, ω_H satisfies the conditions of a norm [2], [3], [4]:

but the condition $\omega_H(\alpha x) = |\alpha|\omega_H(x)$. (1)

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Instead, it satisfies

$\omega_H(\alpha x) = \omega_H(x)$ for all $\alpha \notin 0$.

We may call such a function a mininorm.

Mininorms on Vector spaces

1.1 Definition

A mininorm or weight function on a vector space X over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is a function $\omega : X \to \mathbb{R}$ satisfying the following:

$\omega(x) \geq 0$ for all $x \in X$ and (2)

$\omega(x) = 0$ if and only if $x = 0$

$\omega(\alpha x) = \omega(x)$ for all $x \in \mathbb{K}$, $\alpha \notin 0$ (3)

$\omega(x+y) \leq \omega(x) + \omega(y)$ for all $x, y \in X$. (4)

Here X and mininorm ω are called a mininormed space and is denoted by (X, ω). If the mininorm is ω, then it can be written X instead of (X, ω).

We call $\omega(x)$, the mininorm or weight of x.

Example 1. Let $X = \mathbb{R}^k$ over $\mathbb{R}$ is a mininormed space with the mininorm $\omega = \omega_H$.

Note: The same definition works for $X = \mathbb{C}^k$ over $\mathbb{C}$ as well.

We call this mininorm on $X = \mathbb{R}^k$ or $\mathbb{C}^k$, the Hamming mininorm, or the standard mininorm and denote it by ω_H. Thus, $\omega_H(x)$ = number of non-zero co-ordinates of x.

Remark 1. If we define $\rho_\omega(x, y) = \omega(x-y)$, for $x, y \in X$, then ρ_ω defines a metric on X. Thus, every metric space is a mininormed. ρ_ω is called the metric induced by ω.

This ρ_ω satisfies the conditions

(i) $\rho_\omega(x+z,\ y+z) = \rho_\omega(x, y)$ and

(ii) $\rho_\omega(\alpha x, \alpha y) = \rho_\omega(x, y)$,where $\alpha \neq 0$.

(i) is obvious:

For (ii), consider

$$\begin{aligned} \rho_\omega(\alpha x, \alpha y) &= \omega(\alpha x - \alpha y) \\ &= \omega\,(\alpha(x-y)) \\ &= \omega(x-y) \end{aligned}$$

$$= \rho_\omega(x, y). \tag{5}$$

Note: We shall denote the metric induced by ω_H on $\mathbb{R}^k$ or $\mathbb{C}^k$ by ρ_H. Thus, $\rho_H(x, y)$ = number of non-zero coordinates of $x - y$.

Definition 2. A mininormed space (X, ω) which is complete with respect to the metric induced by ω is called a mini Banach space.

For example, $(\mathbb{R}^n, \omega_H)$ is a mini Banach space.

Note: A proof for this fact is given in the latter part.

Remark 2. Let $x, y \in X$, where (X, ω) is a mininormed space. Then, $\omega(x) = \omega(x-y+y) \leq \omega(x-y) + \omega(y)$

So, $\omega(x) - \omega(y) \leq \omega(x-y)$. (6)

Interchanging x and y, we get,
$\omega(y) - \omega(x) \leq \omega(y - x) = \omega(x - y)$.
That is, $-(\omega(x) - \omega(y)) \leq \omega(x - y)$. (7)
From (6) and (7), imply
$|\omega(x) - \omega(y)| \leq \omega(x - y)$. (8)

2. Basic properties of Mininorms

Proposition 1. Every mininorm ω is a continuous function.

Proof:
Let $x_n \to x$ in X. That is, $\omega(x_n - x) \to 0$. (9)
Now, $|\omega(x_n) - \omega(x)| \leq \omega(x_n - x) \to 0$. From (8) and (9)
So, $\omega(x_n) \to \omega(x)$ and ω is continuous.

Remark 3. Let (X, ω) be a mininormed space, $\omega(\alpha x) \leq \omega(x) \forall x \in X, \alpha \in K$.

Proof:
If $\alpha \neq 0$, then $\omega(\alpha x) = \omega(x)$
If $\alpha = 0$, then, $\omega(\alpha x) = \omega(0) = 0 \leq \omega(x)$.

Theorem 1. Let X be a finite – dimensional Space, and every mininorm on X is bounded function.

Proof:
Let ω be a mininorm on X.
Suppose $\dim(X) = n$. Let $\{x_1, x_2, \dots, x_n\}$ be the basis of X.
Take $x \in X$. Then, there exist $\alpha_1, \alpha_2, \dots, \alpha_n \in K$ such that
$x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$.
(10)
Now $\omega(x) \leq \omega(\alpha_1 x_1) + \omega(\alpha_2 x_2) + \cdots + \omega(\alpha_n x_n)$
$\leq \omega(x_1) + \omega(x_2) + \cdots + \omega(x_n)$, using (Remark 3)
Thus, $\omega(x) \leq \alpha$, were,
$\alpha = \omega(x_1) + \omega(x_2) + \cdots + \omega(x_n)$.

Proposition 2. Let (X, ω) be a mininormed space over K. Suppose (x_n) and (y_n) are finite sequences in X, such that $x_n \to x$ and $y_n \to y$ in X. Then,

(i). $x_n + y_n \to x + y$ and
(ii).$\alpha x_n \to \alpha x, \alpha \in K$

Proof:
$d(x_n + y_n, x + y) = \omega\big((x_n + y_n) - (x + y)\big) \leq \omega(x_n - x) + \omega(y_n - y)$.This proves (i).
Proof of (ii) is also easy.
If $\alpha = 0$, then $\alpha x_n = 0$ for all n and $\alpha x = 0$.
If $\alpha \neq 0$, then $d(\alpha x_n, \alpha x) = \omega(\alpha x_n - \alpha x) = \omega\big(\alpha(x_n - x)\big)$
$= \omega(x_n - x) \to 0.$

Proposition 3. Let ω be a minimorm on X.
Let Y be the subspace of X. Here, $x \in X, y \in Y$ and $\alpha \in K$,$\alpha \neq 0$, we have
$\omega(\alpha x + y) \geq dist(x, Y)$
Here $dist(x, Y)$ is the distance from $x\ to\ Y$ defined by
$dist(x, Y) = \inf\{d(x, y)/y \in Y\}$
$= \inf\{d(x - y)/y \in Y\}$

Proof:
Let $\omega(\alpha x + y) = \omega(\alpha(x + y)/\alpha)) = \omega(x + y/\alpha)$
$\geq \inf\{\omega(x - y)/y \in Y\}$, since $y/\alpha \in Y$
$= dist(x, Y)$.

Remark 4. Let $x \in X$ and Y be the subspace of X, and $dist(\alpha x, Y) = dist(x, Y), for\ any\ \alpha \neq 0$.

Proof:
Let $dist(\alpha x, Y) = \inf\{\omega(\alpha x - y)/y \in Y\}$
$= \inf\{\omega(\alpha(x - y/\alpha))/y \in Y\}$
$= \inf\{\omega(x - y/\alpha)/y \in Y\}$
$= \inf\{\omega(x - y)/y \in Y\}$, since Y is a subspace
$= dist(x, Y)$.

3. Structural Properties of Mininormed Euclidean spaces

Theorem 2. Every Euclidean space with the standard mininorm ω_H is complete.
Proof. Consider any Euclide an space $\mathbb{R}^k$ with the standar d mininorm.
Let $(x_n) = (x_n(1), x_n(2), \cdots, x_n(k))$ be a cauchy sequence in[3,4] $\mathbb{R}^k$.
Then, for every $\epsilon > 0$, it is a positive integer n_0, such that $\omega_H(x_n - x_m) < \epsilon$ for all$n, m \geq n_0$[4].
By choosing $\epsilon = 1$, we get n_0 satisfying $\omega_H(x_n - x_m) < 1$ for all $n, m \geq n_0$.
So, $\omega_H(x_n - x_m) = 0$ for all $n, m \geq n_0$, as w_H is the standard minimum.
Hence $x_n = x_m$ for all $n, m \geq n_0$.
This implies that (x_n) is a constant sequence except for a finite number of terms.

In fact, $(x_n) = \left(x_1, x_2, \cdots, x_{n_0}, x_{n_0}, x_{n_0}, \cdots\right)$
Thus (x_n) is convergent to x_{n_0}. So $\mathbb{R}^k$ is complete.
Let us denote the Euclidean space $\mathbb{R}^k$ with standard mininorm w_H by $\mathbb{R}_H^k$.

Theorem 3. Every subset E of $\mathbb{R}_H^k$ is open.
Proof. For $x \in E$, con sider

$$\begin{aligned} B(x,1) &= \left\{y \in \mathbb{R}_H^k / d_H(x,y) < 1\right\} \\ &= \left\{y \in \mathbb{R}_H^k / d_H(x,y) = 0\right\}, \text{ as there is no } d_H \text{ value between 0 and 1.} \\ &= \{x\} \subset E. \end{aligned}$$

Thus, every point of E is an interior point and hence E is open.

Corollary 1. Every subset of $\mathbb{R}_H^k$ is closed.

Proof. For any subset E of $\mathbb{R}_H^k$, E^c is open, by the above theorem. Hence E is closed.

Remark 5. Every subset of $\mathbb{R}_H^k$ is both open and closed.

Theorem 4. $\mathbb{R}_H^k$ is not connected.

Proof. Let E be any non-empty Proper subset of $\mathbb{R}_H^k$.

Then, both E and E^c are open.

Now, $\mathbb{R}_H^k$ is the union of the disjoint open sets E and E^c.

So, $\mathbb{R}_H^k$ cannot be connected.

Theorem 5. Only finite subsets of $\mathbb{R}_H^k$ are compact, Thus $\mathbb{R}_H^k$ is a mini Banach Space.

Proof. Let E be a compact subset of $\mathbb{R}_H^k$.

By Theorem 3 , every singleton set is open.

So, $S = \{\{x\}/x \in E\}$ is an open covering for E.

Since E is compact, S has a finite sub covering for E, say, $\left\{\{x_1\}, \{x_2\}, \cdots, \{x_n\}\right\}$, as E is compact[4].

Hence $E \subset \{x_1, x_2, \cdots, x_n\}$ and so E is a finite set.

In a normed space $(X, \|\|)$, a closed ball of redius r is defined as

$$B_r = \{x \in X / \| x \| \leq r\}.$$

It is a fact that Balls are convex sets in normed spaces[3].

(A set A is convex if $\alpha x + (1-\alpha)y \in A$ for all $x, y \in A$ and $0 \leq \alpha \leq 1$.)

But this result does not hold in $\mathbb{R}_H^k$.

For an example, consider $\mathbb{R}_H^3$.

Take the ball $B_1 = \left\{x \in \frac{\mathbb{R}^3}{\omega_H(x)} \leq 1\right\}$

Now $x = (1,0,0)$ an d$y = (0,2,0) \in B_1$.

Let $\alpha = \frac{1}{2}$. Then, $\alpha x + (1-\alpha)y = \frac{1}{2}x + \frac{1}{2}y = \left(\frac{1}{2}, 1,0\right)$.

So, $\omega_H(\alpha x + (1-\alpha)y) = 2$.

So, $\alpha x + (1-\alpha)y \notin B_1$.

Thus B_1 is not convex.

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