SOME RESULTS ON 2K- FRAMES IN HILBERT SPACES

N. Kiran Kumar1* and G. Upender Reddy2
1, 2. Department of Mathematics, Mahatma Gandhi University, Nalgonda, T.S, India-508254.
iamkk036@gmail.com[1], upendermathsmgu@gmail.com[2]

Abstract:
In this paper we have given the definitions of 2K- frames and 2K-g-frames, with the help of this we have proved, if $\{f_j\}$ *is K-frame and* $\{g_j\}$ *is a 2K-frame then* $\{f_j + g_j\}$ *is a K-frame for H and also, we give a condition on sum of 2K-g frames is a K-g frame for Hilbert space. Some results on 2K-frames were presented by using an injective closed range operator* $K \in B(H)$. *Mathematics Subject Classification (2010): 42C15.*

Keywords: frame, k-frame, g-frame, 2k-frame, 2-kg-frames.

1. Introduction

Frames in Hilbert spaces were introduced by R.J. Duffin and A.C. Schaffer. Frame theory plays an important role in signal processing, sampling theory, coding and communications and so on. Frames were introduced as a better replacement to orthonormal basis. Frame theory was developed by Peter G. Casazza [8] and O.Christensen [7]. A. Najati and A. Rahimi [1] have developed the generalized frame theory and introduced methods for generating g-frames of a Hilbert space. The notion of K-frames has been introduced by L.Gavruta [6] to study the atomic systems with respect to a bounded linear operator K in Hilbert space H. K-frames are more general than classical frames. In K-frames the lower bound only holds for the elements in the range of K. Dingli Hua and Yongdong Huang [3] are proposed for construction methods for K-g-frames. Results on K-frames have been proved through operator-theoritic results on quotient of bounded operators by G. Ramu and P.Johnson[4]. Sitara Ramesan and K.T Ravindran [9] were presented some results on K- frames where $K \in B(H)$ injective closed range operator.

In this paper we have given the definitions of 2K- frames and 2K-g- frames, with the help of this we have proved, if $\{f_j\}$ is K-frame and $\{g_j\}$ is a 2K-frame then $\{f_j + g_j\}$ is a K-frame for H and also, we give a condition on sum of 2K-g frames is a K-g frame for Hilbert space. Some results on K-frames were presented by using an injective closed range operator $K \in B(H)$.

2. Preliminaries

Definition 2.1. A sequence $\{f_j\}_{j\in J}$ of vectors in a Hilbert space H is called a frame if there exist two constants $0 < A \leq B < \infty$, such that

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$$A\|f\|^2 \le \sum_{j\in J}\left|\left\langle f, f_j\right\rangle\right|^2 \le B\|f\|^2 \ \forall f \in H$$

The above inequality is called a frame inequality. The numbers A and B are called the lower and upper frame bounds respectively. If A=B then $\{f_j\}_{j\in J}$ is called tight frame, if A=B=1 then $\{f_j\}_{j\in J}$ is called normalized tight frame. A synthesis operator T: $l_2 \to$ H is defined as $Te_j = f_j$ where $\{e_j\}$ is an orthonormal basis for l_2. The analysis operator

T^*: H $\to$ l_2 is an adjoint of synthesis operator T and is defined as $T^* f = \sum_{j\in J} < f, f_j > e_j \ \forall f \in H$. A frame operator $S = TT^* : H \to H$ is defined as $Sf = \sum_j < f, f_j > f_j \ \forall f \in H$

Throughout this paper $\{H_j, j \in J\}$ will denote a sequence of Hilbert spaces. Let $L(H, H_j)$ be a collection all bounded linear operators from H to H_j and $\{\Lambda_j \in L(H, H_j) : j \in J\}$.

Definition 2.2. A sequence of operators $\{\Lambda_j\}_{j\in J}$ is said to be g-frame for Hilbert space H with respect to sequence of Hilbert spaces $\{H_j, j \in J\}$, if there exist two constants

$0 < A \le B < \infty$, such that $A\|f\|^2 \le \sum_{j\in J}\|\Lambda_j f\|^2 \le B\|f\|^2 \ \forall f \in H$.

The above inequality is called a g-frame inequality. The numbers A and B are called the lower frame bound and upper frame bound respectively. A g-frame $\{\Lambda_j\}_{j\in J}$ for H is said to be g-tight frame if A = B and g-normalized tight frame for H if A = B = 1.

Definition 2.3. Let $\{\Lambda_j\}_{j\in J}$ be a g-frame for Hilbert space H. A g-frame operator

$S^g : H \to H$ is defined as $S^g f = \sum_{j\in J} \Lambda_j^* \Lambda_j f \ \forall f \in H$.

Definition 2.4. Let $K \in B(H)$. A sequence $\{f_j\}_{j\in J}$ in Hilbert space H is said to be a K-frame for H if there exist two constants $0 < A \le B < \infty$, such that

$$A\|K^* f\|^2 \le \sum_{j\in J}|< f, f_j >|^2 \le B\|f\|^2, \ \forall f \in H.$$

Where A and B are called lower and upper frame bounds for k-frame respectively. If K=I, then K-frames are just ordinary frames.

Definition 2.5. Let $K \in L(H)$ and $\Lambda_j \in L(H, H_j)_{j\in J}$. A sequence of operators $\{\Lambda_j\}_{j\in J}$ is said to be K-g-frame for Hilbert space H with respect to sequence of Hilbert spaces $\{H_j\}_{j\in J}$ if there exist two constants $0 < A \leq B < \infty$, such that

$$A\|K^* f\|^2 \leq \sum_{j\in J}\|\Lambda_j f\|^2 \leq B\|f\|^2, \ \forall\ f \in H. .$$

The above inequality is called a K-g-frame inequality. The numbers A and B are called the lower and upper frame bounds of K-g-frame respectively. When K=I, K-g-frame is a g-frame.

A K-g- frame is said to be tight if there exist a positive constant A such that

$$\sum_{j\in J}\|\Lambda_j f\|^2 = A\|K^* f\|^2, \ \forall\ f \in H.$$

If A=1 then $\{\Lambda_j\}_{j\in J}$ is said to be Parseval tight K-g-frame.

We use following theorem in the proof of our main results.

Theorem 2.6. [11]. Let $T \in B(H)$ is an injective and closed range operator if and only if there exists a constant $c > 0$ such that $c \parallel f \parallel^2 \leq \parallel Tf \parallel^2$, for all $f \in H$.

3. K-Frames

Theorem 3.1. If $\{f_j\}_{j\in J}$ is a frame for $R(K)$, then $\{K^* f_j\}_{j\in J}$ is a frame for H and $\{KK^* f_j\}_{j\in J}$ is a K-frame for H, where $K \in B(H)$ is an injective and closed range operator.

Proof. Let $\{f_j\}_{j\in J}$ be a frame for $R(K)$. Then by the definition we have, there exist constants $A, B > 0$ such that,

$$A \parallel f \parallel^2 \leq \sum_{j=1}^{\infty} |\langle f, f_j\rangle|^2 \leq B \parallel f \parallel^2, \forall\ f \in R(K)$$

For $f \in H \ and\ K \in B(H), Kf \in R(K)$, replace f by Kf in above equation and we get

$$A \parallel Kf \parallel^2 \leq \sum_{j=1}^{\infty} |\langle Kf, f_j\rangle|^2 \leq B \parallel Kf \parallel^2.$$

Since $K \in B(H)$ is an injective and closed range operator, then by the theorem 2.6, there exists $c > 0$ such that$c \parallel f \parallel^2 \leq \parallel Kf \parallel^2$, for all $f \in H$.

Therefore,

$$Ac \parallel f \parallel^2 \leq A \parallel Kf \parallel^2 \leq \sum_{j=1}^{\infty} \left|\langle Kf, f_j\rangle\right|^2 \leq B \parallel Kf \parallel^2 \leq B\alpha^2 \parallel f \parallel^2$$

for all $f \in H$ and for some $\alpha > 0$, i.e.

$$A_1 \parallel f \parallel^2 \leq \sum_{j=1}^{\infty} \left|\langle f, K^* f_j\rangle\right|^2 \leq B_1 \parallel f \parallel^2,$$

for all $f \in H$ where $A_1 = Ac > 0, B_1 = B\alpha^2 > 0$. Therefore, $\{K^* f_j\}_{j\in J}$ is a frame for H and hence $\{KK^* f_j\}_{j\in J}$ is a K-frame for H.

Corollary 3.2. Let $K \in B(H)$ be an injective and closed range operator and $\{f_j\}_{j\in J} \subset H$ be such that $\{(K^{-1})^* f_j\}_{j\in J}$ is a frame for $R(K)$. Then $\{f_j\}_{j\in J}$ is a frame for H.

Proof: By the theorem 3.1, if $\{(K^{-1})^* f_j\}_{j\in J}$ is a frame for $R(K)$ then $\{K^*(K^{-1})^* f_j\}_{j\in J}$ is a frame for H and hence $\{K^*(K^*)^{-1} f_j\}_{j\in J}$ is a frame for R(K)

Therefore $\{f_j\}_{j\in J}$ is a frame for H.

Theorem 3.3. If $\{f_j\}_{j\in J}$ is a K-frame for H and K^* is an injective and closed range operator, then there exist constants $A, B > 0$ such that

$$A\|K^* f\|^2 \leq \sum_{j=1}^{\infty} \left|\langle f, f_j\rangle\right|^2 \leq B\|K^* f\|^2$$

for all $f \in H$.

Proof. Since $\{f_j\}_{j\in J}$ is a K-frame for H, there exist constants $C, D > 0$ such that

$$C\|K^* f\|^2 \leq \sum_{j=1}^{\infty} \left|\langle f, f_j\rangle\right|^2 \leq D \parallel f \parallel^2,$$

for all $f \in H$. $K^* \in B(H)$ is an injective closed range operator, hence there exists a>0 such that

$$a \parallel f \parallel^2 \leq \|K^* f\|^2,$$

for all $f \in H$. Therefore, for all $f \in H$,

$$C\|K^* f\|^2 \leq \sum_{j=1}^{\infty} \left|\langle f, f_j\rangle\right|^2 \leq D \parallel f \parallel^2 \leq (D/\mathrm{a})\|K^* f\|^2$$

$$\Rightarrow A\|K^* f\|^2 \leq \sum_{j=1}^{\infty} \left|\langle f, f_j\rangle\right|^2 \leq B\|K^* f\|^2$$

Where $A = C, B = D/a > 0$.

Corollary 3.4. If $\{f_j\}_{j \in J}$ is a K-frame for, then $\{f_j\}_{j \in J}$ is a frame for H, where K^* is an injective and closed range operator.

Proof: By the theorem 3.3, If $\{f_j\}_{j \in J}$ is a K-frame for H where K^* is an injective and closed range operator.

Then for all $f \in H$ there exist constants $A, B > 0$ such that

$$A\|K^* f\|^2 \leq \sum_{j=1}^{\infty} \left|\langle f, f_j\rangle\right|^2 \leq B\|K^* f\|^2$$

Hence $\{f_j\}_{j \in J}$ is a frame for H.

Definition 3.5. A sequence $\{f_j\}_{j \in J}$ in Hilbert space H is said to be a $2K$-frame for H if there exist $A, B > 0$ such that for all $f \in H$.

$$A\|K^* f\|^2 \leq \sum_{j=1}^{\infty} \left|\langle f, f_j\rangle\right|^2 \leq B\|K^* f\|^2.$$

Theorem 3.6: Let $\{f_j\}_{j \in J}$ be a K-frame for H with bounds A_1, B_1 and $\{g_j\}_{j \in J}$ be a 2 K-frame for H with bounds A_2, B_2 such that $0 < B_2 < A_1$. Then $\{f_j + g_j\}_{j \in J}$ is a K-frame for H with frame bounds $A_1 - B_2$ and $B_1 + B_2\|K^*\|^2$.

Proof. By definition of K-frame and $2K$-frame, we have

$$A_1\|K^* f\|^2 \leq \sum_{j=1}^{\infty} \left|\langle f, f_j\rangle\right|^2 \leq B_1 \parallel f \parallel^2 \qquad \ldots (1)$$

and

$$A_2\|K^*f\|^2 \le \sum_{j=1}^{\infty}\left|\langle f, g_j\rangle\right|^2 \le B_2\|K^*f\|^2 \qquad \dots (2)$$

for all $f \in H$. Consider,

$$\sum_{j=1}^{\infty} \left|\langle f, f_j + g_j\rangle\right|^2 \le \sum_{j=1}^{\infty} \left|\langle f, f_j\rangle\right|^2 + \sum_{j=1}^{\infty} \left|\langle f, g_j\rangle\right|^2$$

$$\le B_1 \| f \|^2 + B_2\|K^*f\|^2$$

$$\le (B_1 + B_2\|K^*\|^2) \| f \|^2$$

$$\sum_{j=1}^{\infty} \left|\langle f, f_j + g_j\rangle\right|^2 \le (B_1 + B_2\|K^*\|^2) \| f \|^2 \qquad \dots (3)$$

for all $f \in H$, Consider,

$$A_1\|K^*f\|^2 \le \sum_{j=1}^{\infty} \left|\langle f, f_j\rangle\right|^2 = \sum_{j=1}^{\infty} \left|\langle f, f_j + g_j - g_j\rangle\right|^2 \qquad \text{by (1)}$$

$$\le \sum_{j=1}^{\infty} \left|\langle f, f_j + g_j\rangle\right|^2 + \sum_{j=1}^{\infty} \left|\langle f, g_j\rangle\right|^2$$

$$\le \sum_{j=1}^{\infty} \left|\langle f, f_j + g_j\rangle\right|^2 + B_2\|K^*f\|^2 \qquad \text{by (2)}$$

This implies that, for all $f \in H$

$$A_1\|K^*f\|^2 \le \sum_{j=1}^{\infty} \left|\langle f, f_j + g_j\rangle\right|^2 + B_2\|K^*f\|^2$$

$$\text{i.e. } \sum_{j=1}^{\infty} \left|\langle f, f_j + g_j\rangle\right|^2 \ge (A_1 - B_2)\|K^*f\|^2 \qquad \dots (4)$$

where $A_1 - B_2 > 0$.

Therefore, from (3) and (4) for all $f \in H$ we have

$$(A_1 - B_2)\|K^*f\|^2 \leq \sum_{j=1}^{\infty} \left|\langle f, f_j + g_j\rangle\right|^2 \leq (B_1 + B_2\|K^*\|^2) \parallel f \parallel^2$$

Hence $\{f_j + g_j\}_{j\in J}$ is a K-frame for H with frame bounds $A_1 - B_2$ and $B_1 + B_2\|K\|^2$.

Theorem 3.7. Let $K \in B(H)$. Suppose $\{f_j\}_{j\in J} \subset H$ is a $2K$-frame for H. If $2 \leq p < \infty$ then

$$C\sum_{n=1}^{\infty}\|f_n\|^2 \leq \sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\left|\langle f_n, f_j\rangle\right|^p \leq D\sum_{n=1}^{\infty}\|f_n\|^2$$

Proof: Given that $\{f_j\}_{j\in J} \subset H$is 2K-frame for H. then by the definition

For all $f \in H$ we have

$$A\|K^*f\|^2 \leq \sum_{n=1}^{\infty} \left|\langle f, f_j\rangle\right|^2 \leq B\|K^*f\|^2$$

take $f = f_n$ for all n=1,2, . . .

we get

$$A\|K^*f_n\|^2 \leq \sum_{n=1}^{\infty} \left|\langle f_n, f_j\rangle\right|^2 \leq B\|K^*f_n\|^2 \qquad \dots (5)$$

for $2 \leq p < \infty \implies \frac{1}{2} \geq \frac{1}{p} > 0 \implies 0 < \frac{2}{p} \leq 1.$

Consider

$$\left[\sum_{j=1}^{\infty}\left|\langle f_n, f_j\rangle\right|^p\right]^{2/p} \leq \sum_{j=1}^{\infty} \left|\langle f_n, f_j\rangle\right|^2$$

$$\leq B\|K^*f_n\|^2 \qquad by\ (5)$$

$$\Rightarrow \sum_{j=1}^{\infty}\left|\langle f_n, f_j\rangle\right|^p \leq B^{\frac{1}{2}}\|K^*f_n\|^{\mathrm{p}}$$

$$\Rightarrow \sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\left|\langle f_n, f_j\rangle\right|^p \leq \mathrm{D}\sum_{n=1}^{\infty}\|K^*f_n\|^{\mathrm{p}} \quad where\ D = B^{\frac{1}{2}} \qquad \dots (6)$$

Consider

$$\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\left|\langle f_n,f_j\rangle\right|^p=\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\left(\left|\langle f_n,f_j\rangle\right|^2\right)^{p/2}$$

$$\geq\sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty}\left|\langle f_n,f_j\rangle\right|^2\right)^{\frac{p}{2}}$$

$$\geq\sum_{n=1}^{\infty}\left(A\|K^*f_n\|^2\right)^{\frac{p}{2}}$$

$$=\sum_{n=1}^{\infty}A^{\frac{p}{2}}\|K^*f_n\|^{\mathrm{p}}$$

$$=A^{\frac{p}{2}}\sum_{n=1}^{\infty}\|K^*f_n\|^{\mathrm{p}}$$

$$=C\sum_{n=1}^{\infty}\|K^*f_n\|^{\mathrm{p}}\quad where\ C=A^{\frac{p}{2}}\qquad \ldots(7)$$

From (6) and (7)

$$C\sum_{n=1}^{\infty}\|f_n\|^2\leq\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\left|\langle f_n,f_j\rangle\right|^p\leq D\sum_{n=1}^{\infty}\|f_n\|^2.$$

4. K-g-Frames

Theorem 4.1. Suppose $\{\Lambda_j\}_{j\in J}$ is a K-g- frame for H where $K^*\in B(H)$ is an injective and closed range operator then there exists a constants C and D such that

$$C\|K^*f\|^2\leq\sum_{j\in J}\left\|\Lambda_j f\right\|^2\leq D\|K^*f\|^2,\forall f\in H.$$

Proof: since $\{\Lambda_j\}_{j\in J}$ is a K-g- frame for H then there exists a constants A and B such that

$$\Rightarrow\qquad A\|K^*f\|^2\leq\sum_{j\in J}\left\|\Lambda_j f\right\|^2\leq B\|f\|^2,\forall f\in H.$$

Since $K^*\in B(H)$ is an injective and closed range operator then there exists a constant $a>0$ such that

$$a\|f\|^2 \leq \|K^*f\|^2 \ \ \forall f \in H$$

Therefore, for each $f \in H$

$$A\|K^*f\|^2 \leq \sum_{j\in J}\left\|\Lambda_j f\right\|^2 \leq B\|K^*f\|^2 \leq \frac{B}{a}\|K^*f\|^2$$

$$C\|K^*f\|^2 \leq \sum_{j\in J}\left\|\Lambda_j f\right\|^2 \leq D\|K^*f\|^2, \qquad where\ A = C\ and\ D = \frac{B}{a} > 0.$$

The definition of a 2K-g-frame will now be provided.

Definition 4.2. Let $K \in B(H)$ and $\Lambda_j \in L(H, H_j)_{j\in J}$. A sequence of operators $\{\Lambda_j\}_{j\in J}$ is said to be 2K-g-frame for Hilbert space H with respect to sequence of Hilbert spaces $\{H_j\}_{j\in J}$ if there exist two constants $0 < A \leq B < \infty$, such that

$$A\|K^*f\|^2 \leq \sum_{j\in J}\left\|\Lambda_j f\right\|^2 \leq B\|K^*f\|^2, \forall f \in H.$$

According to the following theorem, the sum of 2K-g-frames equals a K-g-frame in Hilbert space H.

Theorem 4.3. $\{\Lambda_j\}_{j\in J}$ be a 2K-g-frame for Hilbert space H with frame bounds A_1 , B_1 and $\{\Lambda^{|}{}_j\}_{j\in J}$ be a 2K-g-frame for Hilbert space H with frame bounds $A_2\ and\ B_2$ such that $0 < B_2 < A_1$ then $\{\Lambda_j + \Lambda^{|}{}_j\}_{j\in J}$ is a K-g-frame for H with frame bounds $A_1\ or A_2\ \ and\ B_1 + B_2$.

Proof: By the definition of 2K-g- frame, we have

$$A_1\|K^*f\|^2 \leq \sum_{j\in J}\left\|\Lambda_j f\right\|^2 \leq B_1\|K^*f\|^2, \forall f \in H.$$

And

$$A_2\|K^*f\|^2 \leq \sum_{j\in J}\left\|\Lambda^{|}{}_j f\right\|^2 \leq B_2\|K^*f\|^2, \forall f \in H.$$

Now for all $f \in H$, Consider

$$\sum_{j\in J}\left\|(\Lambda_j + \Lambda^{|}{}_j)f\right\|^2 = \sum_{j\in J}\langle(\Lambda_j + \Lambda^{|}{}_j)f, (\Lambda_j + \Lambda^{|}{}_j)f\rangle$$

$$= \sum_{j\in J}\langle\Lambda_j f + \Lambda^{|}{}_j f, \Lambda_j f + \Lambda^{|}{}_j f\rangle$$

$$= \sum_{j\in J} \langle \Lambda_j f, \Lambda_j f\rangle + \langle \Lambda_j f, \Lambda^{|}{}_j f\rangle + \langle \Lambda^{|}{}_j f, \Lambda_j f\rangle + \langle \Lambda^{|}{}_j f, \Lambda^{|}{}_j f\rangle$$

Since the range of Λ_j and $\Lambda^{|}{}_j$ are orthogonal for all $j \in J$, i.e $\mathcal{R}(\Lambda_j) \perp \mathcal{R}(\Lambda^{|}{}_j)$ for all $j \in J$.

$$= \sum_{j\in J} \langle \Lambda_j f, \Lambda_j f\rangle + \langle \Lambda^{|}{}_j f, \Lambda^{|}{}_j f\rangle$$

$$= \sum_{j\in J} \|\Lambda_j f\|^2 + \|\Lambda^{|}{}_j f\|^2$$

$$= \sum_{j\in J} \|\Lambda_j f\|^2 + \sum_{j\in J} \|\Lambda^{|}{}_j f\|^2$$

$$\leq (B_1 + B_2)\|K^* f\|^2$$

Hence $\forall f \in H$

$$\sum_{j\in\in J} \|(\Lambda_j + \Lambda^{|}{}_j)f\|^2 \leq (B_1 + B_2)\|K^* f\|^2$$

Also $\forall f \in H$

$$A_1\|K^* f\|^2 \leq \sum_{j\in J} \|\Lambda_j f\|^2 \leq \sum_{j\in J} \|\Lambda_j f\|^2 + \sum_{j\in J} \|\Lambda^{|}{}_j f\|^2 = \sum_{j\in J} \|(\Lambda_j + \Lambda^{|}{}_j)f\|^2 .$$

$$A_2\|K^* f\|^2 \leq \sum_{j\in J} \|\Lambda^{|}{}_j f\|^2 \leq \sum_{j\in J} \|\Lambda_j f\|^2 + \sum_{j\in J} \|\Lambda^{|}{}_j f\|^2 = \sum_{j\in J} \|(\Lambda_j + \Lambda^{|}{}_j)f\|^2 .$$

Therefore, $\forall f \in H$

$$A_1\|K^* f\|^2 \leq \sum_{j\in J} \|(\Lambda_j + \Lambda^{|}{}_j)f\|^2 \leq (B_1 + B_2)\|K^* f\|^2$$

Or

$$A_2\|K^* f\|^2 \leq \sum_{j\in J} \|(\Lambda_j + \Lambda^{|}{}_j)f\|^2 \leq (B_1 + B_2)\|K^* f\|^2$$

$\{\Lambda_j + \Lambda^{|}{}_j\}_{j\in J}$ is a K-g-frame for H with frame bounds A_1 $or A_2$ and $B_1 + B_2$.

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