ON NANO FUZZY FORMS OF WEAKLY FUZZY OPEN SETS

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Abstract The purpose of this paper is to define and study certain weak forms of Nano fuzzy open sets namely, Nano fuzzy α-open sets, Nano fuzzy semi open sets and Nano fuzzy pre open sets. Various forms of Nano fuzzy α-open sets and Nano fuzzy α-semi open sets corresponding to different cases of approximations are also derived.
Keywords Nano fuzzy topology, Nano fuzzy open sets, Nano fuzzy interior, Nano fuzzy closure, Nano fuzzy α-open sets, Nano fuzzy semi-open sets, Nano fuzzy pre-open sets.
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1. Introduction:

Topology is the field of Mathematics which is having applications in science and engineering that include camouflage filters, classification, digital image processing, forgery detection, Hausdorff raster spaces, image analysis, microscopy, paleontology, pattern recognition, population dynamics, stem cell biology, topological psychology, and visual merchandising. Thus the detail study of topological spaces is very important as research point of view. As the extension of theory of topological spaces, open/closed sets are its basic units. Njastad [10], Levine [9] and Mashhour et al [2] have introduced the α-open, semi-open and pre-open sets respectively in topological spaces. Since then these concepts have been widely investigated. Later, L. Thivagar et al [8] introduced Nano topological spaces in terms of lower approximation, upper approximation and boundary region of a subset of a universe. In addition to this L. Thivagar and C. Richard [8] introduced Nano α-open sets, Nano semi open sets, Nano pre-open sets and Nano regular open sets on Nano topological spaces. R. Navalakhe et al [11] defined Nano fuzzy topological spaces with respect to a fuzzy subset λ of a universe which is defined in terms of lower and upper approximations of λ and studied Nano fuzzy closure and Nano fuzzy interior of a fuzzy subset. The elements of a Nano fuzzy topological space are called the Nano fuzzy open sets. In this paper different forms of Nano fuzzy open sets such as Nano fuzzy α-open sets, Nano fuzzy semi open sets, and Nano fuzzy pre-open sets are introduced. It will give an extension to the theory of Nano fuzzy topological spaces which will provide the scope of finding its applications in various fields of Science and Engineering.

2. Preliminaries:

Definition 2.1 [8] Let $(U, \tau_R(X))$ be a Nano topological space and $A \subseteq U$. Then A is said to be

1. Nano semi open if $A \subseteq NCl(NInt(A))$.

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2. Nano pre-open if $A \subseteq NInt(NCl(A))$.
3. Nano α-open if $A \subseteq NInt(NCl(NInt(A)))$.

Definition 2.2 [3] Let (X, R) be an approximation space and $X/R = \{\lambda_1, \lambda_{2,\dots,}\lambda_n\}$. Then for any $\lambda \leq F(X)$, where $F(X)$ is set of all fuzzy subsets of X, $\underline{R}(\lambda)$ *and* $\overline{R}(\lambda)$ are the lower and upper approximations of λ with respect to R, respectively and they are the fuzzy sets in X/R. That is, $\underline{R}(\lambda)$, $\overline{R}(\lambda)\colon X/R \to [0,1]$, such that

i. $\underline{R}(\lambda)(\lambda_j) = inf_{y \leq \lambda_j}\lambda(y)$ and

ii. $\overline{R}(\lambda)(\lambda_j) = sup_{y \leq \lambda_j}\lambda(y)$, for all $j = 1, 2, 3, \dots\dots, n$.

Definition 2.3 [1] Let λ be a fuzzy set in a fuzzy topological space (X, τ). Then the fuzzy boundary of λ is defined as

i. $Bd(\lambda) = Cl(\lambda) \wedge Cl(\lambda^c)$

ii. $Bd(\lambda) = Cl(\lambda) - Int(\lambda)$.

Definition 2.4 [4,12] Let λ be a fuzzy subset of X. Then following are equivalent:

i. Upper approximation of λ denoted by $\overline{R}(\lambda)$ is the fuzzy closure operator.

ii. Lower approximation of λ denoted by $\underline{R}(\lambda)$ is the fuzzy interior operator.

Also,

$$Bd(\lambda) = Cl(\lambda) - Int(\lambda) = \bar{R}(\lambda) - \underline{R}(\lambda)$$

2.5 Properties of Fuzzy Approximation Space [12]

Let R be an arbitrary relation from X to Y. The lower and upper approximation operators of a fuzzy set denoted by $\underline{\mathrm{R}}$ and $\overline{\mathrm{R}}$ respectively, satisfies the following properties: for all $\alpha, \beta \in F(X)$,

(FL1) $\underline{R}(\alpha) = (\overline{R}(\alpha^C))^C$

(FU1) $\overline{R}(\alpha) = (\underline{R}(\alpha^C))^C$

(FL2) $\underline{R}(\alpha \wedge \beta) = \underline{R}(\alpha) \wedge \underline{R}(\beta)$

(FU2) $\overline{R}(\alpha \vee \beta) = \overline{R}(\alpha) \vee \overline{R}(\beta)$

(FL3) $\alpha \leq \beta \Rightarrow \underline{R}(\alpha) \leq \underline{R}(\beta)$

(FU3) $\alpha \leq \beta \Rightarrow \overline{R}(\alpha) \leq \overline{R}(\beta)$

(FL4) $\underline{R}(\alpha \vee \beta) = \underline{R}(\alpha) \vee \underline{R}(\beta)$

(FU4) $\overline{R}(\alpha \wedge \beta) = \overline{R}(\alpha) \wedge \overline{R}(\beta)$

Definition 2.6 [11] Let X be a non-empty finite set, R be an equivalence relation on X, $\lambda \leq X$ be a fuzzy subset and $\tau_R(\lambda) = \{1_\lambda, 0_\lambda, \underline{R}(\lambda), \overline{R}(\lambda), Bd(\lambda)\}$. Then by property (2.5), $\tau_R(\lambda)$ satisfies the following axioms

i. $0_\lambda, 1_\lambda \in \tau_{(R)}(\lambda)$ where $0_\lambda\colon \lambda \to I$ denotes the null fuzzy sets and $1_\lambda\colon \lambda \to I$ denotes the whole fuzzy set.

ii. Arbitrary union of members of $\tau_{(R)}(\lambda)$ is a member of $\tau_{(R)}(\lambda)$.

iii. Finite intersection of members of $\tau_{(R)}(\lambda)$ is a member of $\tau_{(R)}(\lambda)$.

That is, $\tau_{(R)}(\lambda)$ is a topology on X called the Nano fuzzy topology on X with respect to λ. We call $(X, \tau_{(R)}(\lambda))$ as the Nano fuzzy topological space (NFTS). The elements of the Nano fuzzy topological space that is $\tau_{(R)}(\lambda)$, are called Nano fuzzy open sets and elements of $[\tau_{(R)}(\lambda)]^C$ are called Nano fuzzy closed sets.

Definition 2.7 [11] The basis for the Nano fuzzy topology $\tau_{(R)}(\lambda)$ with respect to λ is given by $B = \{1_\lambda, \underline{R}(\lambda)(x), \bar{R}(\lambda)(x)\}$.

Definition 2.8 [11] Let $(X, \tau_{(R)}(\lambda))$ be a Nano fuzzy topological space with respect to λ where $\lambda \leq X$ and if $\mu \leq X$ then the Nano fuzzy interior of μ is defined as union of all Nano fuzzy open subsets of μ and it is denoted by $NfInt(\mu)$. That is, it is the largest Nano fuzzy open subset contained in μ.

Similarly, the Nano fuzzy closure of μ is defined as the intersection of all Nano fuzzy closed sets containing μ. It is denoted by $NfCl(\mu)$ and it is the smallest Nano fuzzy closed set containing μ.

3. Methodology:

The concept of fuzzy logic and fuzzy sets was introduced by L. Zadeh [7] in 1968. It was a revolution in the field of Mathematics as research work got new dimensions to recreate different models and structures in fuzzy environment. C.L. Chang [5] had introduced fuzzy topological spaces. L. Thivagar [8] had given the concept of Nano topological spaces and also introduced different topological structures such as Nano closed sets, Nano open sets, Nano closure, Nano interior Nano semi-open sets, Nano pre-open sets, Nano α-open sets etc. The study of all these sets/structures was expansion of the theory of Nano topological spaces.

As the base of Nano topological spaces are rough sets and D. Dubois and H. Prade [6] found the relation between fuzzy sets and rough sets. To define the Nano topological spaces, we need to define lower approximation, upper approximation and boundary region of a subset of a universe. As there is relation between rough sets and fuzzy sets, it seemed possible to fuzzify the Nano topological spaces.

Y. Y. Yao [12] has found in his studies that the fuzzification of a fuzzy subset of a universe can be done in two ways. First is the crisp approximation of a fuzzy subset and second is the fuzzy approximation of a fuzzy subset. B.K. Tripathy and G.K. Panda [3] have defined the crisp approximation spaces for a fuzzy subset of given universe.

Considering this as the base idea, we have defined Nano fuzzy topological spaces [11], in terms of upper approximation, lower approximation and boundary region of a fuzzy subset. This paper is an attempt to define fuzzy α-open sets, Nano fuzzy semi-open sets, Nano fuzzy pre-open sets and their properties.

4. Nano fuzzy α-open sets:

In this paper $(X, \tau_{(R)}(\lambda))$ represents Nano fuzzy topological space with respect to λ where $\lambda \leq X$ (fuzzy subset of X) and R is an equivalence relation on X where X/R denotes the family of equivalence classes of X by R.

Definition 4.1 Let $(X, \tau_{(R)}(\lambda))$ be a Nano fuzzy topological space and $\mu \leq X$. Then μ is said to be

1. Nano fuzzy semi open if $\mu \leq NfCl(NfInt(\mu))$
2. Nano fuzzy pre-open if $\mu \leq NfInt(NfCl(\mu))$
3. Nano fuzzy α-open if $\mu \leq NfInt(NfCl(NfInt(\mu)))$

NFSO (X, λ), NFPO (X, λ) and $\tau_R^\alpha(\lambda)$ respectively denotes the families of all Nano fuzzy semi open, Nano fuzzy pre open and Nano fuzzy α-open subsets of X.

Definition 4.2 Let $(X, \tau_{(R)}(\lambda))$ be a Nano fuzzy topological space and $\mu \leq X$. Then μ is said to be Nano fuzzy α-closed (respectively, Nano fuzzy semi closed, Nano fuzzy pre closed), if its compliment is Nano fuzzy α-open (Nano fuzzy semi open, Nano fuzzy pre-open respectively).

Theorem 4.3 If μ is Nano fuzzy open in $(X, \tau_{(R)}(\lambda))$, then it is Nano fuzzy α-open in X.

Proof: Since μ is Nano fuzzy open in X, $NfInt(\mu) = \mu$. Then $NfCl\big(NfInt(\mu)\big) = NfCl(\mu) \geq \mu$. That is, $\mu \leq NfCl\big(NfInt(\mu)\big)$. Therefore, $NfInt(\mu) \leq NfInt(NfCl(NfInt(\mu)))$. That is, $\mu \leq NfInt(NfCl(NfInt(\mu)))$. Thus, μ is Nano fuzzy α-open.

Theorem 4.4 $\tau_R^\alpha(\lambda) \leq NFSO\ (X, \lambda)$ in a Nano fuzzy topological space $(X, \tau_{(R)}(\lambda))$.

Proof: If $\mu \in \tau_R^\alpha(\lambda)$, $\mu \leq NfInt(NfCl(NfInt(\mu))) \leq NfCl(NfInt(\mu))$ and hence $\mu \leq NSPO\ (X, \lambda)$.

Theorem 4.5 $\tau_R^\alpha(\lambda) \leq NFPO\ (X, \lambda)$ in a Nano fuzzy topological space $(X, \tau_{(R)}(\lambda))$.

Proof: If $\mu \in \tau_R^\alpha(\lambda)$, $\mu \leq NfInt(NfCl(NfInt(\mu)))$. Since $NfInt(\mu) \leq \mu$, $NfInt(NfCl(NfInt(\mu))) \leq NfInt(NfCl(\mu)$. That is, $\mu \leq NfInt(NfCl(\mu))$. Therefore, $\mu \in NFPO(X, \lambda)$. That is, $\tau_R^\alpha(\lambda) \leq NFPO\ (X, \lambda)$.

Theorem 4.6 $\tau_R^\alpha(\lambda) = NFSO(X, \lambda) \Lambda\, NFPO\ (X, \lambda)$.

Proof: If $\mu \in \tau_R^\alpha(\lambda)$, then $\mu \in NFSO(X, \lambda)$ and $\mu \in NFPO(X, \lambda)$ by theorems 4.4 and 4.5 and hence $\mu \in NFSO(X, \lambda) \Lambda\, NFPO\ (X, \lambda)$. That is, $\tau_R^\alpha(\lambda) \leq NFSO(X, \lambda) \Lambda\, NFPO\ (X, \lambda)$. Conversely, if $\mu \in NFSO(X, \lambda) \Lambda\, NFPO\ (X, \lambda)$, then $\mu \leq NfCl(NfInt(\mu))$and $\mu \leq NfInt(NfCl(\mu))$. Therefore, $NfInt\big(NfCl(\mu)\big) \leq NfInt\Big(NfCl\Big(NfCl\big(NfInt(\mu)\big)\Big)\Big) = NfInt\Big(NfCl\big(NfInt(\mu)\big)\Big)$. That is, $NfInt\big(NfCl(\mu)\big) \leq NfInt\Big(NfCl\big(NfInt(\mu)\big)\Big)$. Also, $\mu \leq NfInt\big(NfCl(\mu)\big) \leq NfInt\Big(NfCl\big(NfInt(\mu)\big)\Big)$ implies that $\mu \leq$

$NfInt\left(NfCl\big(NfInt(\mu)\big)\right)$. That is, $\mu \in \tau_R^{\alpha}(\lambda)$. Thus, $NFSO(X,\lambda) \wedge NFPO\,(X,\lambda) \leq \tau_R^{\alpha}(\lambda)$. Therefore, $\tau_R^{\alpha}(\lambda) = NFSO(X,\lambda) \wedge NFPO\,(X,\lambda)$.

Theorem 4.7 If, in a Nano fuzzy topological space $\left(X, \tau_{(R)}(\lambda)\right)$, $\underline{R}(\lambda) = \overline{R}(\lambda) = \lambda$, then 1_λ, 0_λ, $\underline{R}(\lambda)(= \overline{R}(\lambda))$ and any fuzzy set $\mu \geq \underline{R}(\lambda)$ are the only Nano fuzzy α-open sets in X.

Proof: Since $\underline{R}(\lambda) = \overline{R}(\lambda) = \lambda$, the Nano fuzzy topology, $\tau_{(R)}(\lambda) = \{1_\lambda, 0_\lambda, \underline{R}(\lambda)\}$. Since any Nano fuzzy open set is Nano fuzzy α-open, 1_λ, $0_\lambda, \underline{R}(\lambda)$ are Nano fuzzy α-open in X. If $\mu \leq \underline{R}(\lambda)$, then $NfInt(\mu) = 0_\lambda$, since 0_λ is the only Nano fuzzy open subset of μ. Therefore, $NfInt(NfCl(NfInt(\mu))) = 0_\lambda$ and hence μ is not Nano fuzzy α-open. If $\mu \geq \underline{R}(\lambda)$, $\underline{R}(\lambda)$ is the largest Nano fuzzy open subset of μ and hence, $NfInt\left(NfCl\big(NfInt(\mu)\big)\right) = NfInt\left(NfCl\left(\underline{R}(\lambda)\right)\right) = NfInt(Bd(\lambda)^C) = NfInt(1_\lambda)$, since $Bd(\lambda) = 0_\lambda$. Therefore, $NfInt\left(NfCl\big(NfInt(\mu)\big)\right) = 1_\lambda$ and hence, $\mu \leq NfInt\left(NfCl\big(NfInt(\mu)\big)\right)$. This means that μ is Nano fuzzy α-open. Thus 1_λ, $0_\lambda, \underline{R}(\lambda)(= \overline{R}(\lambda))$ and any set $\mu \geq \underline{R}(\lambda)$ are the only Nano fuzzy α-open sets in X, if $\underline{R}(\lambda) = \overline{R}(\lambda)$.

Theorem 4.8 $1_\lambda,\ 0_\lambda\ and\ \overline{R}(\lambda)$ and any set $\mu \geq \overline{R}(\lambda)$ are the only Nano fuzzy α-open sets in Nano fuzzy topological space $\left(X, \tau_{(R)}(\lambda)\right)$ if $\underline{R}(\lambda) = 0_\lambda$.

Proof: Since $\underline{R}(\lambda) = 0_\lambda, Bd(\lambda) = \overline{R}(\lambda)$. Therefore, $\tau_{(R)}(\lambda) = \{1_\lambda, 0_\lambda, \overline{R}(\lambda)\}$ and the members of $\tau_{(R)}(\lambda)$ are Nano fuzzy α-open in X. Let $\mu \leq \overline{R}(\lambda)$. Then $NfInt(\mu) = 0_\lambda$ and hence $NfInt\left(NfCl\big(NfInt(\mu)\big)\right) = 0_\lambda$. Therefore, μ is not Nano fuzzy α-open μ is not Nano fuzzy α-open. If $\mu \geq \overline{R}(\lambda)$, $\overline{R}(\lambda)$ is the largest Nano fuzzy open subset of μ (unless $\overline{R}(\lambda) = 1_\lambda$, in case of which 1_λ and 0_λ are the only Nano fuzzy α-open sets in X). Therefore, $NfInt\left(NfCl\big(NfInt(\mu)\big)\right) = NfInt\left(NfCl\left(\overline{R}(\lambda)\right)\right) = NfInt(1_\lambda)$ and hence $\mu \leq NfInt\left(NfCl\big(NfInt(\mu)\big)\right)$. Thus, any set $\mu \geq \overline{R}(\lambda)$ is Nano fuzzy α-open set in X. Hence, $1_\lambda,\ 0_\lambda\ and\ \overline{R}(\lambda)$ and any superset of $\overline{R}(\lambda)$ are the only Nano fuzzy α-open sets in Nano fuzzy topological space $\left(X, \tau_{(R)}(\lambda)\right)$.

Theorem 4.9 If $\overline{R}(\lambda) = 1_\lambda$ and $\underline{R}(\lambda) \neq 0_\lambda$, in a Nano fuzzy topological space $\left(X, \tau_{(R)}(\lambda)\right)$, then $1_{\lambda,}\ 0_\lambda, \underline{R}(\lambda)\ and\ Bd(\lambda)$ are the only Nano fuzzy α-open sets in X.

Proof: Since $\overline{R}(\lambda) = 1_\lambda$ and $\underline{R}(\lambda) \neq 0_\lambda$, the Nano fuzzy open sets in X are $1_{\lambda,} 0_\lambda, \underline{R}(\lambda)\ and\ Bd(\lambda)$ and hence they are Nano fuzzy α-open also. If $\mu = 0_\lambda$, then μ is Nano fuzzy α-open. Therefore, let $\mu \neq 0_\lambda$. When $\mu \leq \underline{R}(\lambda), NfInt(\mu) = 0_\lambda$, since the largest fuzzy open subset of μ is 0_λ and hence $\mu \nleq NfInt\left(NfCl\big(NfInt(\mu)\big)\right)$, unless μ is 0_λ. That is, μ is not Nano fuzzy α-open in X. When $\underline{R}(\lambda) \leq \mu$, $NfInt(\mu) = \underline{R}(\lambda)$ and therefore, $NfInt\left(NfCl(NfInt(\mu))\right) = NfInt\left(NfCl\left(\underline{R}(\lambda)\right)\right) = NfInt(Bd(\lambda)^C) = NfInt\left(\underline{R}(\lambda)\right) = \underline{R}(\lambda) \leq \mu$. That is, $\mu \nleq NfInt\left(NfCl(NfInt(\mu))\right)$. Therefore, μ is not Nano fuzzy α-open in X. Similarly, it can be shown that any fuzzy set $\mu \leq Bd(\lambda)$ and $\mu \geq Bd(\lambda)$ are not Nano fuzzy α-open in X. If μ has at least one element each of $\underline{R}(\lambda)$ and $Bd(\lambda)$, then $NfInt(\mu) = 0_\lambda$ and hence μ is not Nano fuzzy α-open in X. Hence, $1_\lambda, 0_\lambda, \underline{R}(\lambda)$ *and* $Bd(\lambda)$ are the only Nano fuzzy α-open sets in X when $\overline{R}(\lambda) = 1_\lambda$ and $\underline{R}(\lambda) \neq 0_\lambda$.

Corollary 4.10 $\tau_{(R)}(\lambda) = \tau_R^\alpha(\lambda)$, if $\overline{R}(\lambda) = 1_\lambda$.

Theorem 4.11 Let $\underline{R}(\lambda) \neq \overline{R}(\lambda)$ where $\underline{R}(\lambda) \neq 0_\lambda$ and $\overline{R}(\lambda) = 1_\lambda$ in a Nano fuzzy topological space $\left(X, \tau_{(R)}(\lambda)\right)$. Then $1_\lambda, 0_\lambda, \underline{R}(\lambda), Bd(\lambda)$, $\overline{R}(\lambda)$ and any fuzzy set $\mu \geq \overline{R}(\lambda)$ are the only Nano fuzzy α-open sets in X.

Proof: The Nano fuzzy topology on X is given by $\tau_R(\lambda) = \left\{1_\lambda, 0_\lambda, \underline{R}(\lambda), \overline{R}(\lambda), Bd(\lambda)\right\}$ and hence $1_\lambda, 0_\lambda, \underline{R}(\lambda), \overline{R}(\lambda)$ *and* $Bd(\lambda)$ are Nano fuzzy α-open sets in X. Let $\mu \leq \lambda$ such that $\mu \geq \overline{R}(\lambda)$. Then $NfInt(\mu) = \overline{R}(\lambda)$ and therefore, $NfInt\left(NfCl\left(\overline{R}(\lambda)\right)\right) = NfInt(1_\lambda) = 1_\lambda$. Hence, $\mu \leq NfInt\left(NfCl(NfInt(\mu))\right)$. Therefore, any $\mu \geq \overline{R}(\lambda)$ is Nano fuzzy α-open in X. When $\mu \leq \underline{R}(\lambda), NfInt(\mu) = 0_\lambda$, and hence $NfInt\left(NfCl(NfInt(\mu))\right) = 0_\lambda$. Therefore, μ is not Nano fuzzy α-open in X. When $\mu \leq Bd(\lambda)$, $NfInt(\mu) = 0_\lambda$ and hence μ is not Nano fuzzy α-open in X. When $\mu \leq \overline{R}(\lambda)$ such that μ is neither a subset of $\underline{R}(\lambda)$ nor a subset of $Bd(\lambda)$, $NfInt(\mu) = 0_\lambda$ and hence μ is not Nano fuzzy α-open in X. Thus, $1_\lambda, 0_\lambda, \underline{R}(\lambda), Bd(\lambda)$, $\overline{R}(\lambda)$ and any fuzzy set $\mu \geq \overline{R}(\lambda)$ are the only Nano fuzzy α-open sets in X.

5. Forms of Nano fuzzy semi-open sets

In this section, we have tried to derive forms of Nano fuzzy semi-open sets depending on various combinations of approximations.

Remark 5.1 $1_\lambda, 0_\lambda$ are obviously Nano fuzzy semi-open sets, since $NfCl(NfInt(1_\lambda)) = 1_\lambda$ and $NfCl(NfInt(0_\lambda)) = 0_\lambda$.

Theorem 5.2 If, in a Nano fuzzy topological space $\left(X, \tau_{(R)}(\lambda)\right)$, $\underline{R}(\lambda) = \overline{R}(\lambda)$, then 0_λ and fuzzy sets μ such that $\mu \geq \underline{R}(\lambda)$ are the only Nano fuzzy semi-open subsets of X.

Proof: In Nano fuzzy topological space $\tau_R(\lambda) = \{1_\lambda, 0_\lambda, \underline{R}(\lambda)\}$, 0_λ is obviously Nano fuzzy semi-open. If μ is a non-empty fuzzy subset of X and $\mu \leq \underline{R}(\lambda)$, then $NfCl(NfInt(\mu)) = NfCl(0_\lambda) = 0_\lambda$. Therefore, μ is not Nano fuzzy semi-open, if $\mu \leq \underline{R}(\lambda)$. If $\mu \geq \underline{R}(\lambda)$, then $NfCl(NfInt(\mu)) = NfCl\left(\underline{R}(\lambda)\right) = 1_\lambda$, since $\underline{R}(\lambda) = \overline{R}(\lambda)$. Therefore, $\mu \leq NfCl(NfInt(\mu))$ and hence μ is Nano fuzzy semi-open.

Theorem 5.3 If $\underline{R}(\lambda) = 0_\lambda$ and $\overline{R}(\lambda) \neq 1_\lambda$, then only those fuzzy sets containing $\overline{R}(\lambda)$ are the Nano fuzzy semi-open sets in X.

Proof: In this case Nano fuzzy topology will be $\tau_R(\lambda) = \{1_\lambda, 0_\lambda, \overline{R}(\lambda)\}$. Let μ be a non-empty fuzzy subset of X. If $\mu \leq \overline{R}(\lambda)$, then $NfCl(NfInt(\mu)) = NfCl(0_\lambda) = 0_\lambda$ and hence $\mu \nleq NfCl(NfInt(\mu))$. Therefore, μ is not Nano fuzzy semi-open in X. If $\mu \geq \overline{R}(\lambda)$, then $NfCl(NfInt(\mu)) = NfCl\left(\overline{R}(\lambda)\right) = 1_\lambda$ and hence $\mu \leq NfCl(NfInt(\mu))$. Therefore, μ is Nano fuzzy semi-open in X.

Theorem 5.4 If $\overline{R}(\lambda) = 1_\lambda$ in a Nano fuzzy topological space then $1_\lambda, 0_\lambda, \underline{R}(\lambda)$ and $Bd(\lambda)$ are the only Nano fuzzy semi-open sets in X.

Proof: If $\overline{R}(\lambda) = 1_\lambda$, Nano fuzzy topological space will be $\tau_R(\lambda) = \{1_\lambda, 0_\lambda, , \underline{R}(\lambda), Bd(\lambda)\}$. Let μ be a non-empty fuzzy subset of X. If $\mu \leq \underline{R}(\lambda)$, then $NfCl(NfInt(\mu)) = 0_\lambda$ and hence μ is not Nano fuzzy semi-open in X. If $\mu = \underline{R}(\lambda)$, then $NfCl(NfInt(\mu)) = NfCl\left(\underline{R}(\lambda)\right) = \underline{R}(\lambda)$ and hence $\mu \leq NfCl(NfInt(\mu))$.Therefore, μ is Nano fuzzy semi-open in X.If $\mu \geq \underline{R}(\lambda)$, then $NfCl(NfInt(\mu)) = NfCl\left(\underline{R}(\lambda)\right) = \underline{R}(\lambda)$. Therefore, $\mu \nleq NfCl(NfInt(\mu))$ and hence μ is not Nano fuzzy semi-open in X.If $\mu \leq Bd(\lambda)$, $NfCl(NfInt(\mu)) = 0_\lambda$ and hence μ is not Nano fuzzy semi-open in X. If $\mu = Bd(\lambda)$, then $NfCl(NfInt(\mu)) = NfCl(Bd(\lambda)) = Bd(\lambda)$ and hence $\mu \leq NfCl(NfInt(\mu))$. Therefore, μ is Nano fuzzy semi-open in$X$.If $\mu \geq Bd(\lambda)$, then $NfCl(NfInt(\mu)) = NfCl(Bd(\lambda)) = Bd(\lambda) \leq \mu$ and hence μ is not Nano fuzzy semi-open in X. Thus $1_\lambda, 0_\lambda, \underline{R}(\lambda)$ and $Bd(\lambda)$ are the only Nano fuzzy semi-open sets in X if $\overline{R}(\lambda) = 1_\lambda$ and $\underline{R}(\lambda) \neq 0_\lambda$. If $\underline{R}(\lambda) = 0_\lambda$, 1_λ and 0_λ are the only Nano fuzzy semi-open sets in X, since $1_\lambda$and 0_λ are the only sets in X which are Nano fuzzy open and Nano fuzzy closed.

Theorem 5.5 If μ and γ are Nano fuzzy semi-open sets in X, then $\mu \vee \gamma$ is also Nano fuzzy semi-open in X.

Proof: If μ and γ are Nano fuzzy semi-open sets in X, then $\mu \leq NfCl(NfInt(\mu))$ and $\gamma \leq NfCl(NfInt(\gamma))$. Consider $\mu \vee \gamma \leq NfCl(NfInt(\mu)) \vee NfCl(NfInt(\gamma)) = NfCl(NfInt(\mu) \vee NfInt(\gamma)) = NfCl(NfInt(\mu \vee \gamma))$ and hence $\mu \vee \gamma$ is Nano fuzzy semi-open in X.

Conclusion: Earlier, when the theory of Nano topological spaces was introduced, it was expanded by defining different topological structures on them. One of them was weak forms of Nano open sets. When we have defined and introduced a new type of fuzzy topological spaces which are called Nano fuzzy topological spaces, we are also trying to find different fuzzy topological structures by which we will be able to develop and expand the theory of Nano fuzzy topological spaces. After the development of theory of Nano topological spaces, it's applications were found. So this paper can be a small step towards development of theory and finding the different applications of Nano fuzzy topological spaces.

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