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A COMMON FIXED POINT THEOREM IN I-GENERALIZED METRIC SPACES

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Abstract

Here in, a common fixed point theorem of two self-mapping in the frame of generalized metric space under more generalized $(\psi - \phi)$-weakly condition of contraction is presented, and then, an analogous version of this common fixed point theorem is presented in the frame of I-generalized metric space.

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1. Introduction

In 1986, G. Jungck [6] proved common fixed point theorem under the concept of compatible self- mapping in metric spaces.

In 1997, Alber and Delabrier [1] introduced the concept of φ-weak contraction under which the existence of fixed points for self-mapping of Hilbert spaces has been proved.

Definition of a φ-weak contraction is "A mapping $T: X \to X$ is called a φ-weak contraction if there exists a continuous and non-decreasing function $\varphi: [0,\infty) \to [0,\infty)$ such that $\phi(t) = 0$ iff $t = 0$ for which, for all $x, y \in X$, $d(T(x), T(y)) \le d(x, y) - \varphi\, d(x, y)$".

In fact, Banach contraction appears to be a special case of weak contraction by taking $\varphi(t) = (1-\alpha)t$ for $0 \le \alpha < 1$.

In 2001, Rhoades [9] proved the result of Alber and Delabrieer in complete metric spaces.

In 2008 Dutta and Choudhury [5] introduced the concept of $(\psi - \phi)$-contraction and proved the fixed point result of self-mapping in metric spaces stated as "Let (X, d) be a complete metric space and

$T : X \to X$ satisfy $\psi(d(T(x),T(y))) \le \psi(d(x, y)) - \varphi(d(x, y))$ for all $x,y \in X$, where $\psi,\varphi : [0,\infty) \to [0,\infty)$ are continuous non-decreasing functions such that $\psi(t)=0=\varphi(t)$ iff $t=0$. Then T has a unique fixed point". Taking $\psi(t) = t$ ($t \ge 0$), we get the φ-weak contraction. Taking $\psi(t)=t$ ($t\ge0$) and $\varphi(t)=(1-\alpha)t$ with $0<\alpha<1$, we get Banach contraction.

In 2009, Doric [4] proved common fixed point result of two self-mapping under $(\psi - \phi)$-contraction stated as "Let (X,d) be a complete metric space and $T,f:X\to X$ satisfy

$\psi(d(T(x), f(y))) \le \psi(M(x, y)) - \varphi(M(x, y))$ for all $x,y \in X$, where $M(x,y) = max\left\{d(x,y), d\big(x,T(x)\big), d\big(y,f(y)\big), \frac{1}{2}\big(d\big(x,f(y)\big) + d\big(y,T(x)\big)\big)\right\}$, and

(i) $\psi:[0,\infty)\to[0,\infty)$ is continuous non-decreasing such that $\psi(t)=0$ iff $t=0$,

(ii) $\varphi:[0,\infty)\to[0,\infty)$ is lower semi-continuous such that $\varphi(t)=0$ iff $t=0$.

Then T and f have a unique fixed point in X.

In 2000 Branciari [3] introduced metric space, called generalized metric space, which is a generalization of traditional metric space replacing triangular inequality by rectangular

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inequality and thereafter many fixed point theorems and common fixed point theorems have been proved in this frame.

In [10], I-generalized metric space has been introduced, which is a kind of generalization of generalized metric space, and some fixed point results under several contractions in the context of
I-generalized metric spaces have been proved.

In [2] existence of unique common fixed point of two self- mapping of rectangular metric spaces under $(\psi - \phi)$-weakly contractive condition has been established, stated as " Let (X, d) be a Housdorff rectangular metric space, $S, T: X \rightarrow X$ such that $S(X) \subset T(X)$ and $(T(X), d)$ is a complete rectangular metric space, and satisfy $\psi\left(d\big(S(x), S(y)\big)\right) \leq \psi\left(M\big(T(x), T(y)\big)\right) - \phi(M(T(x), T(y)))$ for all $x, y \in X$, and ψ, ϕ are continuous with $\psi(t) = 0$ iff $t = 0$, $\phi(t) = 0$ iff $t = 0$ and ψ is non-decreasing, and $M\big(T(x), T(y)\big) = max\big\{d\big(T(x), T(y)\big), d\big(T(x), S(x)\big), d(T(y), S(y))\big\}$. Then *S*, *T* have a unique coincidence point in *X*. Moreover, if *S* and *T* are weakly compatible, then *S*, *T* have a unique common fixed point".

Here we are weakening the $(\psi - \phi)$-weakly contraction of [2] and establish new common fixed point result of two self-mapping of g.m.s., and then prove the analogous version of this result in the I-g. m. s.

2. Preliminaries

First we remind some notation and definitions that will be utilized in our subsequent discussion.

Definition (2.1) [I-uniqueness or I-equality]: [10] Let *X* be a non-empty set and $f: X \rightarrow X$ be an idempotent map. Two elements x and y in X are said to be I-unique with respect to *f*, if $f(x) = f(y)$; otherwise *x* and *y* are said to be I-distinct points in *X*.

Definition (2.2)[I-generalized metric space]: [10] Let *X* be a non-empty set, $f: X \rightarrow X$ be an idempotent map, *i.e.*, $f^2 = f$. A map $d: X^2 \rightarrow [0, \infty)$ is said to be an I-generalized metric (I-g.m.s., in short) on *X* iff

I_1: $\forall x, y \in X, d\big(x, f(y)\big) = 0 \ iff \ f(x) = f(y)$ and $d(f(x), y) = 0 \ iff \ f(x) = f(y)$.

I_2: $d\big(x, f(y)\big) = d(y, f(x))$ and $d(f(x), y) = d(f(y), x), \forall x, y \in X$.

I_3: for all $x, y \in X$ and for all I-distinct points $u, v \in X$ each of which I-distinct from *x* and *y*, $d(x, y) \leq d(f(x), u) + d(f(u), v) + d(v, f(y))$.

The order triple $(X, d, f)$is called an I-generalized metric space. Elements of *X* are said to be points in *X*.

Example (2.3): (i) Every I-metric space is clearly a I-g.m.s.

(ii) Every generalized metric space (*X*,*d*) is clearly a I-g-m-s. with respect to the identity map on *X*.

(iii) Let $X = A \cup B$, where $A = \{0, 2\}, B = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Let $f: X \to X$ be an idempotent mapping. Define $d: X^2 \to [0, \infty)$ by

$$d(x,y) = \begin{cases} 0, if\ f(x) = f(y) \\ 1, if\ f(x) \neq f(y), \{f(x), f(y)\} \subseteq A\ or\ \{f(x), f(y)\} \subseteq B \\ f(y), if\ f(x) \in A, f(y) \in B \\ f(x), if\ f(x) \in B, f(y) \in A \end{cases}$$

Then (X, d, f) is an I-g.m.s.

Definition (2.4) [Convergence of a sequence:] [10] A sequence $\{x_n\}$ in an I-g.m.s. (X, d, f) is said to I-converge to a point $x \in X$, if for any $\varepsilon > 0, \exists m \in \mathbb{N}$ such that $d(f(x_n), x) < \varepsilon, \forall n \geq m$. In this case x is called I-limit of $\{x_n\}$.

A sequence which is not I-convergent in an I-g.m.s. (X,d, f), is called a non-I-convergent or an I-divergent sequence.

Definition (2.5) [Cauchy sequence]:[10] A sequence $\{x_n\}$ in an I-g.m.s. (X, d, f) is said to be an I-cauchy sequence in X if for any $\varepsilon > 0, \exists n_o \in \mathbb{N}$ such that $d(f(x_m), x_n) < \varepsilon, \forall m, n \geq n_o$, *i.e.*, $d(f(x_{n+p}), x_n) < \varepsilon, \forall n \geq n_o, \forall p \geq 1$.

Definition (2.6) [Complete I-g.m. s.]: [10] An I-g.m.s. (X,d,f) is said to be I-complete if every I-cauchy sequence in X I-converges to some point of X ;otherwise (X,d,f) is called I-incomplete.

Definition (2.7)[I-fixed point]: [10] Let X be a non-empty set and $f: X \to X$ is an idempotent map. A map $h: X \to X$ is said to have an I-fixed point $x (\in X)$ if $(fh)(x) = f(x)$.

Theorem (2.8): [10] Let (X,d,f) be an I-g.m.s. Then

(i) $d(x, x) = 0, \forall x \in X, i.e., \forall x, y \in X, x = y \Longrightarrow d(x, y) = 0$.

(ii) $d(x, f(y)) = d(f(x), y) = d(f(x), f(y)) = d(f(y), f(x)) \geq d(x, y), d(y, x), \forall x, y \in X$.

(iii) $d(x, f(x)) = 0, \forall x \in X$.

Proof: Trivial.

Definition (2.9)[Coincidence point]: Let X be a non-empty set and $S, T: X \to X$. A point $x \in X$ is called a coincidence point of S and T if $S(x) = T(x)$ and if $S(x) = T(x) = w$, then w is called a point of coincidence of S and T .

Also S and T are said to be weakly compatible if $(ST)(x) = (TS)(x)$ whenever $S(x) = T(x)$.

3. Main Results

Theorem (3.1) Let (X, d) be a g.m.s. and $S, T: X \to X$ such that $(X) \subset T(X)$. Let $(T(X), d)$ be a complete g.m.s.

Let $\psi(d(S(x), S(y))) \leq \psi(M(T(x), T(x))) - \phi(M(T(x), T(y))), \forall x, y \in X$ (1)

Where ψ, ϕ are continuous with $\psi(t) = 0$ iff $t = 0$, $\phi(t) = 0$ iff $t = 0$ and ψ is non-decreasing, and $M(T(x), T(y)) = max\{d(T(x), T(y)), d(T(x), S(x)), d(T(y), S(y)), d(T(y), S(x)), d(S(x), S(y))\}$. Then S, T have a unique point of coincidence in X. Moreover, if S and T are weakly compatible, then S, T have a unique common fixed point.

Proof: Let u and v be points of coincidence of S and T in X. then there exists x, y in X, Such that $\mathrm{S}(x) = \mathrm{T}(x) = u$ and $\mathrm{S}(y) = \mathrm{T}(y) = v$ (2)
Now from (1) we get,

$$\psi(d(u,v)) = \psi(d(s(x),S(y))) \leq \psi(M(T(x),T(y))) - \phi(M(T(x),T(y))). \quad (3)$$

where

$$M(T(x),T(y)) = \max. M(d(\mathrm{u},v),d(u,u),d(v,v),d(v,u),d(u,v) = d(u,v))$$

Therefore (3) becomes

$$\psi(d(u,v)) \leq \psi(d(u,v) - \phi(d(u,v)) < \psi(d(u,v)) < \psi(d(u,v)\ if d(u,v) > 0$$

which is not possible. therefore $d(u,v) = 0$. Therefore $u = v$.

Therefore, If S and T have one point of coincidence, there it is unique.

Let u be the point of coincidenoc of S and T.then there exists $x \in X$. Such that $S(x) = T(x) = u$ (4)

Since S and T are weakly compatible, we have $(ST)(x) = (TS)(x)$

$\Rightarrow\ S(u) = T(u) = u$ (say) (5)

Since S and T have unique point of coincidence, from (4) and (5) we have $u = v$

Therefore $\mathrm{S}(u) = u = T(u)$ (6)

So that u is a common fixed point of S and T. Let w be any common fixed point of S and T.

Then $S(w) = w = T(w)$ (7)

$\Rightarrow\ w$ is a point of coincidence of S and T. Since S and T have a unique point of concidence, from (6) and (7) we have $u = w$. Therefore S and T have unique common fixed point. If S and T have a common fixed point. Then it is unique.
Let $x_0 \in X$ be arbitrary since $S(x)$ C $\mathrm{T}(x)$. Define two scquences $\{x_n\},\{y_n\}$ in X as follows:

$$y_0 = S(x_0) = T(x_1)\ (\text{ Since }\ j_0 = S(x_0)CS(x) \subset T(x))$$

There exist $x_1 \in X$ such that $S(x_0) = T(x_1))$

$$y_1 = S(x_1) = T(x_2)$$

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Therefore, $\forall n \in N \cup \{0\},\ y_n = S(x_n) = T(x_{n+1})$

If $y_{\mathrm{n}} = y_{n-1}$ for some $n \in N$, Then $y_{n-1} = S(x_{n-1})$

$\Rightarrow\ T(x_n) = y_n = S(x_n)$ So that y_{n-1} is the coincidence point of S and T.

Let $y_0 \neq y_{n-1},\ \forall n \in N$

If $y_n = S(x_n) = S(x_{n+1}) = (y_{n+1})$ for some $n \geq 0, p \geq 2$ then

$$y_n = S(x_n) = T(x_{n+1}) = S\left(x_{n+p}\right) = T\left(x_{n+p+1}\right) = y_{n+p}$$

So that, at time of construction of the sequence $\{x_n\}$ and $\{y_n\}$, we choose $x_{n+p+1} = x_{n+1}, \forall n \geq 0$

In this case, from (1) we get,

$$\begin{aligned}&\psi\big(d(y_n, y_{n+1})\big) = \psi\left(d\big(s(x_n), S(x_{n+1})\big)\right)\\ \leq\ &\psi\left(M\big(T(x_n), T(x_{n+1})\big) = \phi\left(M\big(T(x_n), T(x_{n+1})\big)\right)\right.\end{aligned} \quad (8)$$

Where $M(T(x_n), T(x_{n+1}) = \mathrm{Max}\,\{d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1}), d(y_3) = \mathrm{Max}\,\{d(y_{n+1}, y_n), d(y_n, y_{n+1})\}$

If $M\big(T(x_n), T(x_{n+1})\big) = d(y_n, y_{n+1})$, Then from (8)

We have $\psi\big(d(y_n, y_{n+1})\big) \leq \psi\left(d(y_n, y_{n+1}), \psi\big(d(y_n, y_{n+1})\big) \leq \psi\big(d(y_n, y_{n+1})\big) - \psi\big(d(y_n, y_{m+1})\big)\right)$

$\Rightarrow\ \phi(d(y_n, y_{n+1}) = 0 \Rightarrow d(y_n, y_{n+1}) = 0$

$\Rightarrow\ y_n = y_{n+1}$ which is not true.

Therefore (8) becomes,

$$\begin{aligned}&\psi\big(d(y_n, y_{n+1})\big) \leq \psi\big(d(y_{n-1}, y_n)\big) - \phi\big(d(y_{n-1}, y_n)\big) \leq \psi\big(d(y_{n-1}, y_n)\big)\\ \Rightarrow\ &d(y_n, y_{n+1}) \leq d(y_{n-1}, y_n), \forall n \in N \text{ (since } \psi\ is\ non-decreasing)\end{aligned}$$
(9)

Therefore $\{d(y_n, y_{n+1})\}$ is a decreasing sequence of non-negative sequence of real numbers so that it wis converges to some real number $r(\geq 0)$. Let $r > 0$

Since ϕ, ψ are continuous, from (9) we get

$\psi(r) \leq \psi(r) - \phi(r) < \psi(r)$ since $r > 0 \Rightarrow \phi(r) < 0$ a contradiction. Therefore r = 0 therefore $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$
(10)

Now from (1) we get,

$$\Psi\big(d(y_n, y_{n+1})\big) = \psi\big(d\big(s(x_n)\big), S(x_{n+2})$$
$$\leq \quad \psi\Big(M\big(T(x_n), T(x_{n+2})\big) - \phi\Big(M\big(T(x_n), T(x_{n+2})\big)\Big) \tag{11}$$

Therefore $\{d(y_n, y_{n+1})\}$ is a decreasing sequence of non-negative real number so that it converges to some real number $r(\geq 0)$. Let $r > 0$.

Since ψ, ϕ are continuous, from (8) we get $\psi(r) \leq \psi(r) - \phi(r) < \psi(r)$

(Since $> 0 \Rightarrow \phi(r) > 0$) a contradiction. Therefore $r = 0$ Therefore

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$$

Now from (1) we get,

$$\Psi\big(d(y_n, y_{n+1})\big) = \psi\big(dS(x_n)S(x_{n+1})\big)$$
$$\leq \quad \psi\big(M\big(T(x_n), T(x_{n+2})\big) - \phi\big(M\big(T(x_n), T(x_{n+2})\big)$$

where $M\big(T(x_n), T(x_{n+2})\big) = \max\{d(y_{n+1}, y_{n-1})\}, d(y_{n-1}, y_n), d(y_{n+1}, y_{n+2}),$

$$d(y_{n+1}, y_n), d(y_n, y_{n+2})\}$$

$\Rightarrow \ \max\{d(y_{n+1}, y_{n+1}), d(y_{n-1}, y_n), d(y_n, y_{n+1})\},$

[Since $d(y_{n+1}, y_{n+2}) \leq d(y_n, y_{n+1}) \leq d(y_{n-1}, y_n)$].

Let $M\big(T(x_n), T(x_{n+1})\big) = d(y_{n-1}, y_{n+1})$ Then (11) becomes

$\psi\big(d(y_{n-1}, y_n)\big) = \psi\big(d(y_{n-1}, y_{n+1})\big) - \phi(d(y_{n-1}, y_{n+1}) \qquad \leq \psi\big(d(y_{n-1}, y_{n+1})\big)$
(12)

$\Rightarrow \ d(y_n, y_{n+2}) \leq d(y_{n-1}, y_{n+1}), \forall n \in N$ in that $\{d(y_n, y_{n+2})\}$ is a decreasing sequence of non-negative real numbers and hence it connerges to some real number $r(\geq 0)$

$$\text{If } y_n = S(x_n) = T(x_{n+1}) = S\big(x_{n+p}\big) = T\big(x_{n+p+1}\big)$$

So that, at the time of construction of the sequence $\{x_n\}$ and $\{y_n\}$, we choose

$x_{n+p+1} = x_{n+1}, \forall n \geq 0$. in this case from (1), we get,

$$\psi(d(y_n, y_{n+1})) = \psi\left(d(S(x_n), S(x_{n+1}))\right)$$
$$\leq \psi\left(M(T(x_n), T(x_{n+1}))\right) - \phi\left(M(T(x_n), T(x_{n+1}))\right)$$

Where $M(T(x_n)), T(x_{n+1}) = \max\{d(y_{n+1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1}), d(y_n, y_{n+1})\}$

$= \text{m } \{d(y_{n-1}, y_n), d(y_n, y_{n+1})\}$

If $M(T(x_n), T(x_{n+1})) = d(y_n, y_{n+1})$, then from (7) we get,

$$\psi\left(d(y_n, y_{n+1}) \leq \psi(d(y_n, y_{n+1}))\right) - \psi(d(y_n, y_{n+1}))$$
$$\Rightarrow \phi(d(y_n, y_{n+1})) = 0 \Rightarrow d(y_n, y_{n+1}) = 0$$
$\Rightarrow y_n = y_{n+1}$ which is not the case.

Therefore $M(T(x_n), T(x_{n+1})) = d(y_{n+1}, y_n)$

Therefore (8) becomes, $\psi(d(y_n, y_{n+1})) \leq \psi(d(y_{n-1}, y_n) - \phi(d(y_{n-1}, y_n))$
$\leq \psi(d(y_{n-1}, y_n))$

$\Rightarrow d(y_n, y_{n+1}) \leq d(y_{n-1}, y_n), \forall n \in N$ (Since, ψ is non-decreasing).

Where $M(T(x_n), T(x_{n+2}))$

$= \text{m } ax.\{d(y_{n-1}, y_{n+1}), d(y_{n-1}, y_n), d(y_{n+1}, y_{n+2}), d(y_{n+1}, y_n), d(y_n, y_{n+2})\}$
$= \text{m } ax.\{d(y_{n-1}, y_{n+1}), d(y_{n-1}, y_n), d(y_n, y_{n+2})\}$
{ since $d(y_{n+1}, y_{n+2}) \leq d(y_n, y_{n+1}) \leq d(y_{n-1}, y_n)$}

Let $M(T(x_n), T(x_{n+2})) = d(y_{n-1}, y_{n+1})$ Then (10) becomes

$$\psi(d(y_n, y_{n+2})) \leq \psi(d(y_{n-1}, y_{n+1})) - \phi(d(y_n, y_{n+1}))$$
$$\leq \psi(d(y_{n-1}, y_{n+1}))$$

$\Rightarrow d(y_n, y_{n+1}) \leq d(y_{n-1}, y_{n+1}), \forall n \in N$ in that $[d(y_n, y_{n+2})]$ is a decreasing sequence of Non-negative real numbers and hence it converges to some real number and hence it converges to some real number $r (\geq 0)$

Let $r > 0$ Now by continuing of ϕ, ψ from (12) we get

$\psi(r) = \psi(r) - \phi(r) \leq \psi(r)$ (Since $r > 0 \Rightarrow 0(r) > 0$ a contradiction.

Therefore $r = 0$ therefore $\lim_{n\to\infty} d(y_n, y_{n+2}) = 0$ in this case

Let $M\big(T(x_n),T(x_{n+2})\big)=d(y_{n-1},y_n)$, then (12) becomes.

$\psi\big(d(y_n,y_{n+2})\big)\le\psi\big(d(y_{n-1},y_n)\big)-\phi\big(d(y_{n-1},y_n)\big)$.

Let $\psi\big(T(x_n)\big),T(x_{n+1})\Big)=d(y_{n-1},y_n)$ Then (12) becomes

$\psi\big(d(y_n,y_{n+2})\le\psi\big(d(y_{n-1},y_n)\big)-\phi\big(d(y_{n-1},y_n)\big)$

$\Rightarrow\ 0\le\lim_{n\to\infty}\psi\big(d(y_n,y_{n+2})\big)\le\psi(0)-\phi(0)=0$

(by (10), continuity of ϕ,ψ and since $\phi(0)=0=\psi(0)$)

$\Rightarrow\ \lim_{n\to\infty}\psi\big(d(y_n,y_{n+2})\big)=0$, in this case

$\Rightarrow\ \psi\left(\lim_{n\to\infty}d(y_n,y_{n+2})\right)=0$ (by property of ψ)

Let $M\big(T(x_n),T(x_{n+2})\big)=d(y_n,y_{n+2})$. Then (11) becomes

$\psi\big(d(y_n,y_{n+2})\big)\le\psi\big(d(y_n,y_{n+2})\big)-\phi\big((y_n,y_{n+2})\big)$

$\Rightarrow\ \phi\big(d(y_n,y_{n-2})\big)=0\Rightarrow d(y_n,y_{n+2})=0$ (by property of ϕ)

$\Rightarrow\ y_n=y_{n+2}$ which is not case.

Therefore, in any admissible case, we have $\lim_{n\to\infty}d(y_n,y_{n+2})=0$ (13)

Let $\lim_{n\to\infty}d\big(y_n,y_{n+p}\big)\le d\big(y_n,y_{n+p-1}\big)+d\big(y_{n+p-1},y_{n+p}\big)+d\big(y_{n+p},y_{n+p+1}\big)\to 0$

by (13) and (10) Therefore

$\lim_{n\to\infty}d\big(y_n,y_{n+p+1}\big)=0$. Therefore by mathematical induction,

$\lim_{n\to\infty}d\big(y_n,y_{n+p}\big)=0$ for any integer $p\ge 1$. Therefore $\{y_n\}$ is cauchy in X.

Now for all $n\in N\cup\{0\},y_n=S(x_n)=T(x_{n+1})\in T(X)$

Therefore $\{y_n\}$ is couchy in $\{T(x),d\}$, So that $\{y_n\}$ converges to some point $u\in T(x)$.

Now $u\in T(x)\Rightarrow u=T(v)$ for some $v\in XM\big(T(x_n)\big),T(U)$

$$\lim_{n\to\infty}M(T(x_n),T(v))=\max.\{0,0,d(u,s(v)),0,d(u,s(v))\}=d(u,s(v))$$

from (1) we get $\psi\big(d(s(x_n),s(v))\big)\le\psi\big(M(T(x_n),T(v))\big)-\phi(M(T(x_n),T(v))$ (14)

Where $M(T(x_n), T(v)) = \max\{d(y_{n-1}, u), d(y_{n-1}, y_n), d(u, s(v)), d(u, y_n), d(y_n, s(v))\}$ (15)

Now $d(y_n, s(v)) \leq d(y_n, y_{n-1}) + d(y_{n-1}, u) + d(u, s(v))$

$\leq d(y_{n-1}, y_n) + d(y_{n-1}, u) + d(u, y_{n-1}) + d(y_{n-1}, y_n) + d(y_n, s(v))$

$\Rightarrow \lim_{n\to\infty} d(y_n, s(v)) \leq 0 + 0 + d(u, s(v)) \leq 0 + 0 + 0 + 0 + \lim_{n\to\infty} d(y_n, s(v))$

$(by(10)$ and $\lim_{n\to\infty} y_n = u)$

$$\Rightarrow u = s(v) \Rightarrow T(v) = s(v) \quad (16)$$

from (16) we get,

$$\lim_{n\to\infty} M(T(x_n), T(v)) = \max.\{0,0, d(u, S(v),0, d(u, S(v))\} = d(u, S(v)) \quad (17)$$

(by $(10), (16)$), Therefore from (14) we get

$\lim_{n\to\infty} \psi(d(y_n, S(v)) \leq \psi(d(u, S(v))) - \phi(d(u, S(v))$ …. (by continuity of ϕ, Ψ)

$\Rightarrow 0 \leq \psi(d(u, S(v)) \leq \psi(d(u, S(v)) - \phi(d(u, S(v))$ (by continity of ψ and 16))

$\Rightarrow \phi(d(u, S(v)) = 0 \Rightarrow d(u, S(v)) = 0$ (by property of ϕ) $\Rightarrow u = S(v) \Rightarrow T(v) = S(v)$

Let $w = T(v) = S(u)$, then w is a point of coincidence of S and T and hence it is unique.(already proved)

Let S and T be weak compatible. Then we have already proved w is a unique. Common fixed point of S and T

Definition (3.2) Let $f, S, T: X \to X$, where X is a nonemply set and f is idempotent.

(a) A point $x \in X$ is called an I-coincidence point of S and T (with respect of) If $(fs)(x)$

$= (fT)(x)$ and if $(fs)(x) = (fT)(x) = w$ then w is called a point of I-coincidence of S and T.

(b) S and T are said to be weakly I-compatible. If $(fSfT)(x) = (fTfS)(x)$ whenever $(fS)(x) = (fT)(x)$

Theorem (3.3) Let (X, d, f) be an I-g.m.s. Let $S, T; X \to X$ such that $S(X) \subset T(X)$.

Let $(T(x), d, f)$ is an I-complete I-g.m.s. provided $(fT)(x) \subset T(x)$.

Let $\psi(d(fs)(x), s(y)) \le \psi(M((fT)(x), T(y))) - \phi(M((fT)(x), T(y))), \forall x, y \in X$
(1)

where $\psi, \phi \in \psi$ nondecreasing and $M((fT)(x), T(y))$
$= \max.\{d((fT)(x), T(y)), d((fT), (x), S(s))$
$d((fT)(y), S(y)), d((fT)(y), S(x)), d((fS)(x), S(y))\}$

Then S and T have an I-unique point of I-unique common I-fixed point.

Proof: Let u and v be points of I-coincidence of S and T. then there exists x, y in X Such that $(fS)(x) = (fT)(x) = u$ and $(fS)(y) = (fT)(y) = v$
(2)

Now from (1) we get,

$\psi(d(f(u), v)) = (d(f(u), f(v)) = \psi(d((fS)(x), (fS)(y)))$ (by (2))
$= \psi(d(fS(x), S(y))) \le \psi(M((fT)(x) \cdot T(y))) - \phi(M)((fT))(x), T(y)))$
(3)

Where $M((fT)(x), T(y))) =$
max$\{d(f(u), v), d(f(u), v)$,$d(f(u), v), d(f(v), u), d(f(u, v)) = d(f(u), v)$

Therefore (3) becomes

$$\psi(d(f(u), v)) \le \psi(d(f(u), v) - \phi(d(f(v), v))$$
$$< \psi(d(f(u), v)), \text{ If } d(f(f(u), v) > 0$$

Which is not possible therefore $d(f(u), v) = 0$

Therefore $f(u), = f(v)$. Therefore S and T have a point of I-coincidence then it is I-unique.

Let u be the I-unique point of I-coincidence of S and T. then there exists x in X such that

$(fS)(x) = (fT)(x) = u$ (4)

Since S and T are weakly I-compatible,

We have $(fSfT)(x) = (fTfS)(x)$
$\Rightarrow (fS)(u) = (fT)(u) = v$ (say) (5)

Since S and T have I-unique point of I-coincidence then from (4) and (5), $f(u) = f(v)$

Therefore $(fS)(u) = f(u) = f(T)(u)$ (6)

As u is the common fixed point of S and T.

Let w be any common I-fixed point of S and T. Then $(f(x)(x) = f(w) = (fT)(w)$

Then $(fS)(w) = f(w) = (fT)(w)$ (7)

$f(w)$ is a point of I-coincidence of S and T.
S and T have I-unique point of coincidence,

Therefore S and T have a I-unique commen fixed point. Therefore, If S and T have a commen Lined point. Let $x_0 \in X$ be arbitrary, since $S(x) \subset T(x)$ define two sequences $\{x_n\}, \{y_n\}$ in X as follows $y_0 = S(x_0) = T(x_1)$ Since $y_0 = S(x_0) \in S(x) \subset T(x)$, there exist $x, \in X$ such that $S(x_0) = T(x_1)$

$y_1 = S(x_1) = T(x_2)$…and so on

Hence, $\forall n \in N \cup \{0\}, \forall_n = S(x_n) = T(x_{n+1})$

$$f(y_n) = f(y_{n-1}) \text{ for some } n \in N, \text{ then}$$
$$f(y_{n-1}) = (f\ S)(x_{n-1}) = (fT)(x_n) = f(y_n) = (f\ S)(x_n)$$

So that $f(y_{n-1})$ is the I-unique point of coinidence of S and T, or x_n is the I-unique I-coincidence point
and T of S and T

Let $f(y_n) \neq f(y_{n-1}), \forall n \in N$.
If $f(y_0) = (fS)(x_n) = (fS)(x_{n+p}) = f(y_{n+p})$

for some $n \geq 0, p \geq 2$ then

$$\begin{aligned} f(y_n) &= (fS)(x_n) = (fT)(x_{n+1}) = (fS)(x_{n+p}) = (fT)(x_{n+u+1}) \\ &= f(y_{n+u}) \text{ so that at the time of construction of the sequence} \end{aligned}$$

We choose $x_{n+p+1} = x_{n+1}, \forall n \geq 0$, in this case, from (1) we get

$$\begin{aligned} \psi(d(f(y_n), y_{n+1})) &= \psi\left(d((fS)(x_n), S(x_{n+1}))\right) \\ &\leq \psi\left(M((fT)(x_n), T(x_{n+1}))\right) - \varphi(M((fT(x_n), T(x_{n+1})) \end{aligned} \quad (8)$$

Where $M((fT)(x_n), T(x_{n+1}))$.

$$= \max.\{d(f(y_{n-1}), y_n), d(f(y_{n-1}), y_n), d(f(y_n), y_{n+1}), \\ d(f(y_n), y_n), d(f(y_n), (y_{n+1}))\}$$

If $M((fT)(x_n), T(x_{n+1}) = d(f(y_n), y_{n+1})$, then from (8) we get,

$$\psi\big(d(f(y_n), y_{n+1})\big) \leq \psi\big(d(f(y_n), y_{n+1})\big) - \phi\big(d(f(y_n), y_{n+1})\big)$$
$$\Rightarrow \quad \psi\big(d(f(y_n), y_{n+1})\big) = 0 \Rightarrow d(f(y_n), y_{n+1}) = 0$$
$$\Rightarrow \quad f(y_n) = f(y_{n+1}) \text{ which is not the case.}$$

Therefore $M\big((fT)(x_n), T(x_{n+1})\big) = d(f(y_{n-1}), y_n)$

Therefore (8) becomes.

$$\psi\big(d(f(y_n), y_{n+1})\big) \leq \psi(d(y_{n-1}), y_n) - \phi\big(d(f(y_{n-1}), y_n)\big) \leq \psi\big(d(f(y_{n-1}), y_n)\big) \qquad (9)$$

$$\Rightarrow \; d(f(y_n), y_{n+1}) \leq d(f(y_{n-1}), y_n), \forall n \in N \text{ (since } \psi \text{ is non decreasing.)}$$

Therefore $\{d(f(y_n), y_{n+1})\}$ is a decreasing sequence of non-negative real number so that it comerges to some real number $r(\geq 0)$ a contradiction therefore $r = 0$

Therefore $\lim_{n\to\infty} d(f(y_n), y_{n+1}) = 0$ (10)

Now from (1) we get,

$$\psi\big(d(f(y_n), y_{n+2})\big) = \psi\big(d(fS)(x_n), S(x_{n+2})\big)$$
$$\leq \psi\Big(M\big((fT)(x_n), T(x_{n+2})\big)\Big) - \phi\Big(M\big((fT)(x_n), T(x_{n+2})\big)\Big) \qquad (11)$$

Where $M\big((fT)(x_n), T(x_{n+2})\big)$

$$= \max.\{d(f(y_{n-1}), y_{n+1}), d(f(y_{n-1}), y_n)$$
$$d(f(y_{n+1}), y_{n+2}), d(f(y_{n+1}), y_n), d(f(y_n), y_{n+2})\}$$
$$= \max.\{d(f(y_{n-1}), y_{n+1}), d(f(y_{n-1})y_n), d(f(y_n), y_{n+2})\}$$
$$\big(\text{ since } d(f(y_{n+1}), y_{n+2}) \leq d(f(y_n), y_{n+1}) \leq d\big(f(y_{n-1}, y_n)\big)$$
$$\text{Let } M\big((fT)(x_n), T(x_{n+2})\big) = d(f(y_{n-1}), y_{n+1}) - \phi\big(d(f(y_{n-1}), y_{n+1})\big)$$
$$\leq \psi\big(d(f(y_{n-1}), y_{n+1})\big)$$
$$\Rightarrow \; d(f(y_n), y_{n+2}) \leq d(f(y_{n-1}), y_{n+1}), \forall n \in N \qquad (12)$$

So that $\{d(f(y_n), y_{n\in 2})\}$ is a decreasing sequence of nonege to some real number $r(\geq 0)$.

Let $r > 0$ Now by continuity of ϕ, ψ, from (12) we get

$\psi(r) \leq \psi(r) - \phi(r) < \psi(r)$ (since, $r > 0 \Rightarrow \phi(r) > 0$) a contradiction therefore $r = 0$

$\lim_{n\to\infty} d(f(y_n), y_{n+2}) = 0$ in this case. Let $\lim_{n\to\infty} d(f(y_n), y_{n+2}) = 0$ (13)

Then (12) becomes,

$$\psi\big(d(f(y_n)\cdot y_{n+2})\big) \le \psi\big(d(f(y_{n-1}),y_n)\big) - \phi\big(d(f(y_{n-1}),y_n)\big)$$
$$\Rightarrow \quad 0 \le \lim_{n\to\infty} \psi\big(d(f(y_n),y_{n+2})\big) \le \psi(0)-\phi(0) = 0$$

(by (10), continuity of ϕ,ψ and since $\phi(0) = 0 = \psi(0)$

$$\Rightarrow \lim_{n\to\infty}\big(d(f(y_n),y_{n+2})\big) = 0 \text{ in this case}$$
$$\Rightarrow \psi\lim_{n\to\infty}\big(d(f(y_n),y_{n+2})\big) = 0 \text{ (by continiuty of } \psi \text{)}$$
$$\Rightarrow \lim_{n\to\infty} d(f(y_n),y_{n+2}) = 0 \text{ (by property of } \psi \text{)}$$

Let $M\big((Ft)(x_n),T(x_{n+2})\big) = d(f(y_n),y_{n+2})$

Then (10) becomes.

$$\psi\big(d(f(y_n),y_{n+2}) \le \psi\big(d(f(y_n),y_{n+2})\big) - \phi\big(d(f(y_0),y_{n+1})\big)$$
$$\Rightarrow \quad (\phi(d(f(y_n \ne y_n(2))f0T)(v) = (fS)(v).$$
$$\Rightarrow \quad d(f(y_n),y_{n+2}) = 0 \text{ (by property of } \phi)$$
$$\Rightarrow \quad f(y_n) = f(y_{n+2}) \text{ which is not the case,}$$

We have $\lim_{n\to\infty} d(f(y_n),y_{n+2}) = 0$

Let $\lim_{n\to\infty} d\big(f(y_n),y_{n+p}\big) = 0$, for any integer $p \ge 2$

Now $d\big(f(y_n),y_{n+p+1}\big) \le d\big(f(y_n),y_{n+p-1}\big) + d\big(f\big(y_{n+p-1},y_{n+p}\big) + d\big(f\big(y_{n+p}\big),y_{n+p+1}\big) \to a$ as $n \to \infty$ (by (12), (10)), Therefore $\lim_{n\to\infty} d\big(f(y_n),y_{n+p+1}\big) = 0$

Therefore, by mathematical induction.

$\lim_{n\to\infty} d\big(f(y_n),y_{n+p}\big) = 0$, for any integer $p \ge 1$

Therefore $\{y_n\}$ is I-cauchy in X. Now for $n \in N \cup \{0\}, y_n = \phi S(x_n) = T(x_{n+1}) \in T(x)$ Therefore $\{y_n\}$ is clearly I-Cauchy in $(T(x),d,f)$. So that $\{y_n\}$ I-converges to some point $u \in T$
Now $u \in T(x) \Rightarrow u = T(v)$ for some $v \in X$

from (1) we get,

$$\psi(d(fS)(x_n),S(v))\big) \le \psi(M((fT) \quad (x_n),T(v))\big)$$
$$-\phi(M(fT)(x_n),T(v))\big)$$

Where $M((fT)(x_n), T(v)) = \max\{d(f(y_{n-1}), v), d(f(y_{n-1}), y_n) d(f(u), S(i)), d(f(u), y_n), d(f(y_n), S(u))\big)$

Now $d(f(y_n), S(u)) \leq d(f(y_n), y_{n-1}) + d(f(y_{n-1}), u) + d(f(u), s(v))\}$

$$\leq d(f(y_{n-1}), y_n) + d(f(y_{n-1}, u) + d(f(u), y_{n-1}) + d(f(y_{n-1}), y_n) + d(f(y_n), S(v))$$

$$\lim_{n\to\infty} d(f(y_n), S(v)) \leq 0 + 0 + d(f(u), S(v)) \leq 0 + 0 + 0 + 0 + \lim_{n\to\infty} d(f(y_n), S(v))\Big)$$

(by (10) and $\lim_{n\to\infty} y_n = u$)

$\Rightarrow \lim_{n\to\infty} d(f(y_n), S(v))\big) = d(f(u), S(v))$

from (14), we get

$$\lim_{n\to\infty} M(f(x_n), T(v))\Big) = \max.\{0,0, d(f(u), S(v)),0, d(f(u), S(v))\}$$
$$= d(f(u), S(v))$$

by (10), (13), Therefore from (13) we get,

$$\lim_{n\to\infty} \psi(d(f(g_n), S(v)) \leq \psi(d(f(u), S(v)) - \phi(df(u), S9v)))$$

(by continuity of ϕ, ψ))

$\Rightarrow 0 \leq \psi(d(f(u), S(v)) \leq \psi(d(f(u), S(v)) - \phi(d(f(u), S(v)))$ (by property of ϕ)

$$\Rightarrow \quad f(u) = (fS)(v)$$
$$\Rightarrow \quad (fT)(v) = (fS)(u)$$

Let $W = (fT)(v) = (fS)(v)$.

Then clearly $f(w) = (fT)(v) = (fS)(v)$.

Then w is a point of I-coincidence of S and T and hence it is I-unique (already proved). Let S and T be weak I-compatible. Then we have already proved that w is an I-unique common I-fixed point of S and T.

<u>Conclusion:</u>

Further study may be continued for generalization and extension of various contractive conditions and fixed point results.

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