JOURNAL OF NORTHEASTERN UNIVERSITY

Volume 25 Issue 04, 2022 ISSN: 1005-3026 https://dbdxxb.cn/ Original Research Paper

COMPARISON OF DIFFERENTIAL EQUATIONS AND DIFFERENCE EQUATIONS

Haeder Abdolrazzaq Jasem
Postgraduate Student Affairs Department, Aliraqia University

Abstract
With the help of some conditions for ordinary differential equations, we will demonstrate the fundamental ideas of differential equations and the process by which first-degree difference equations were created. We will also compare how ordinary differential equations are derived with how difference equations are solved, demonstrate some similarities, and prove some properties. We will also demonstrate how to resolve the first degree difference equation.

1-Introduction

Given the importance of difference equations and their wide uses in various fields, where the need for such equations has become urgent to facilitate work and obtain quick results, we will explain in this research the method of derivation of such equations and give examples of them. We want to calculate the annual interest, which is 6% for a fixed amount over 15 years, we will do the following:

Assuming that the amount is 15,000 dinars, and after a year, the result will be 1st year:

$15000 \times 0.06 = 900 \quad , \quad 900 + 15000 = 15900$

The following table shows the results after 15 years.

But by using differential equations we can get the result faster:

$f(x_{15}) = E^{15}f(x_0) = (1.06)^{15} \times 15000 = 35948.3729$

N	amount	rate	interest	new amount
1	15000	0.06	900	15900
2	15900	0.06	954	16854
3	16854	0.06	1011.24	17865.24
4	17865.24	0.06	1071.9144	18937.1544
5	18937.1544	0.06	1136.229264	20073.38366
6	20073.38366	0.06	1204.40302	21277.78668
7	21277.78668	0.06	1276.667201	22554.45388
8	22554.45388	0.06	1353.267233	23907.72112
9	23907.72112	0.06	1434.463267	25342.18439
10	25342.18439	0.06	1520.531063	26862.71545
11	26862.71545	0.06	1611.762927	28474.47838
12	28474.47838	0.06	1708.468703	30182.94708
13	30182.94708	0.06	1810.976825	31993.9239
14	31993.9239	0.06	1919.635434	33913.55934
15	33913.55934	0.06	2034.81356	35948.3729

Submitted: 08/12/2022 **Accepted**: 27/12/2022

Table 1: The stages of increasing the amount over time

2-Calculation of difference equations of the first order

We know from previous studies that differential equations deal with continuous periods, while difference equations deal with discrete periods. In general,
the difference equation is used to solve Ordinary differential equation by finite difference method. Therefore, it is possible to define the difference equation as The relationship between differences that occur in one or more general values of independent variable.
The general form of difference equations is
$Ef(x_n) = f(x_{n+1})$ (1)

We will show how to model this equation using the general definition of the derivative any function. If $f(x)$ is defined on interval [a, b] , and

$$\lim_{\substack{h\to\infty \\ h\neq 0}} \frac{\mathrm{f(x+h)-f(x)}}{h} \quad , \quad x, x+h \in [a,b] \; (2)$$

Then, the function has a derivative at the point x.
There are several symbols to express the derivative $\frac{d}{dx}f(x), f'(x), Df(x)$.

- If the value of $h = 1$ in Equation (2)

$$\frac{f(x+1)-f(x)}{h} = f(x+1) - f(x) (3)$$

$$\Delta f(x) = f(x + 1) - f(x) (4)$$

- Where h ≠1 So there is normalized variable $\bar{x} = \frac{x}{h}$ and define the function

$$\bar{f}\left(\frac{x}{h}\right) = \frac{1}{h} f(x)$$

Then

$$\frac{1}{\mathrm{h}}\mathrm{f(x+h)} = \bar{\mathrm{f}}\left(\frac{\mathrm{x+h}}{\mathrm{h}}\right) = \bar{\mathrm{f}}\left(\frac{\mathrm{x}}{\mathrm{h}}+1\right) = \bar{\mathrm{f}}(\bar{\mathrm{x}}+1)$$

It is clear that

$$\frac{f(x+h)-f(x)}{h} = \frac{f(x+h)}{h} - \frac{f(x)}{h}$$

$$\frac{1}{h}f(x+h) - \frac{1}{h}f(x) = \bar{f}\left(\frac{x+h}{h}\right) - \bar{f}\left(\frac{x}{h}\right)$$

$$\bar{f}(\bar{x}+1) - \bar{f}(\bar{x}) = \Delta\bar{f}(\bar{x})$$

Where (Δ) may be considered as the difference operator applied to the function $f(x)$.
Now we can use (4) to find a second relationship, as follows
$f(x + 1) = \Delta f(x) + f(x)$

$f(x+1) = f(x)(\Delta + 1)$

Assuming that there is a validation relationship

$Ef(x_n) = f(x_{n+1})$

$\Delta f(x_n) = f(x_{n+1}) - f(x_n)$

$\Delta f(x_n) = (E - 1) - f(x_n)$

$E = (1 + \Delta)$, $\Delta = (E - 1)$

These operator may be applied successively

$E^2 f(x_n) = E(Ef(x_n)) = Ef(x_{n+1}) = f(x_{n+2})$

$E^k f(x_n) = f(x_{n+k})$ (5)

$$(1+\Delta)^k f(x_n) = \sum_{j=0}^{k} \binom{k}{j} \Delta^j f(x_n)$$

And

$$\Delta^2 f(x_n) = \Delta(\Delta f(x_n))$$
$$= \Delta(f(x_{n+1}) - f(x_n)) = \Delta(f(x_{n+1})) - \Delta(f(x_n))$$
$$= f(x_{n+2}) - f(x_{n+1}) - (f(x_{n+1}) - f(x_n))$$
$$= f(x_{n+2}) - 2f(x_{n+1}) + f(x_n)$$
$$\Delta^k f(x_n) = (E-1)^k f(x_n)$$
$$= \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} E^j f(x_n)$$
$$= \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x_{n+j})$$

Example 2.1

$f(x_{n+1}) = \alpha f(x_n)$, $f(x_0) = 10$

$f(x_1) = \alpha f(x_0) = \alpha 10$

$f(x_2) = \alpha f(x_1) = \alpha(\alpha f(x_0)) = \alpha^2 10$

using the equation (5)

$f(x_n) = \alpha^n 10$

3-Difference equations of the first order

In the table, we will show a comparison between Differential Equations and Difference Equations:

N	Differential	Difference
1	$Dc = 0, c = a\ constant$	$\Delta c = 0, c = a\ constant$
2	$Dx^n = nx^{n-1}, n = an\ integer$	$\Delta x^n = nx^{n-1}, n = +integer$
3	$Dcf(x) = cDf(x), c = a\ \ constant$	$\Delta cf(x) = c\Delta f(x), c = a\ \ constant$
4	$D[f(x) + g(x)] = Df(x) + Dg(x)$	$\Delta[f(x) + g(x)] = \Delta f(x) + \Delta g(x)$

5	$D[f(x)g(x)] = Df(x)g(x) + Dg(x)f(x)$	$\Delta[f(x)g(x)] = \Delta f(x)g(x) + \Delta g(x)f(x+1)$
6	$D\frac{f(x)}{g(x)} = \frac{g(x)Df(x) - f(x)Dg(x)}{g^2(x)}$	$\Delta\frac{f(x)}{g(x)} = \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x)g(x+1)}$

Table 2: comparison between Differential and Difference quations

Proof (1) : *If* $f(x) = c$, c = constant
$\Delta f(x_n) = f(x_{n+1}) - f(x_n) = c - c = 0$

Proof (2) : $f(x) = x^n$, n = + integer
$\Delta f(x) = f(x + 1) - f(x) = (x + 1)^n - x^n$
The proof is done using the binomial theorem

$$\Delta x^n = \left(x^n + nx^{n-1} + \binom{n}{2}x^{n-2} + \cdots + 1\right) - x^n$$

$$\Delta x^n = \left(nx^{(n-1)} + \binom{n}{2}x^{n-2} + \cdots + 1\right)$$

$$\Delta x^n = \sum_{j=0}^{n-1}\binom{n}{j}x^j \text{ , } n = +integer (6)$$

As a special case, we will define $f(x)$ as a factorial function. To prove the second case we define (x^n) as

$$x^n = x(x-1)(x-2)\ldots(x-n+2)(x-n+1)$$

$$\Delta f(x) = f(x+1) - f(x) = (x+1)^n - x^n$$

$$= \left((x+1)[x(x-1)(x-2)\ldots(x-n+2)]\right) - \left([x(x-1)(x-2)\ldots(x-n+2)](x-n+1)\right)$$

Take the common factor

$$= (x+1) - (x-n+1)[x(x-1)(x-2)\ldots(x-n+2)]$$

$$= n[x(x-1)(x-2)\ldots(x-n+2)]$$

$$= nx^{n-1}$$

Proof: (5)
$\Delta[f(x)g(x)] = f(x+1)g(x+1) - g(x)f(x)$
We add and subtract $f(x+1)g(x)$ we get

$$\Delta[f(x)g(x)] = f(x+1)g(x+1) - f(x+1)g(x) + f(x+1)g(x) - g(x)f(x)$$

$$\Delta[f(x)g(x)] = f(x+1)[g(x+1) - g(x)] + g(x)[f(x+1) - f(x)]$$

$$\Delta[f(x)g(x)] = \Delta f(x)g(x) + \Delta g(x)f(x+1)$$

4-Applications on difference equations

We know from our previous studies the method of solving the differential equa- tion, where we integrate the differential equation several times and according to the rank of the equation to get the original equation. We also use this method with the difference equations and by calculating the rank of the difference equation and according to the law:

$$\frac{The\ largest\ argument - The\ smallest\ argument}{Unit\ of\ increment}$$

e.g $\Delta^2 f(x_n) + \Delta f(x_n) + f(x_n) = n^2$

by using equation (4) we get

$$f(x_{n+2}) - f(x_{n+1}) + f(x_n) = n^2$$

by using (7) we get

$$\frac{(n+2) - n}{1} = 2$$

e.g. The equation $f(x_{n+2}) - 2f(x_{n+1}) + 2f(x_n) = 2^n$ is of order 2

The solution of a difference equation The process of finding a differential equation from its general solution is by deriving the general solution by the number of arbitrary constants present in it. Likewise, in the differential equations, it can be found through the general solution to it, so the rank of the difference equation is equal to the number of arbitrary constants.

For example, if c_1 and c_2 are the arbitrary constants of period 1,

$y_n = c_1 3^n + c_2(-1)^n$

is the general solution of the equation

$f(x_{n+2}) - 2f(x_{n+1}) - 3f(x_n) = 0$

and $y_n = 2(3)^n + 5(-1)^n$ is a particular solution with $c_1 = 2$ and $c_2 = 5$.

5-Forming Difference Equations

We can form the difference equation by removing arbitrary constants from the relationship so that the order of the equation is equal to the number of arbitrary constants.

Example 5.1 : from the relation $\boldsymbol{y_n = c_1 3^n + c_2(-1)^n}$ Form the difference equation.

Solution: Take n as $(n + 1)$ and $(n + 2)$ in the relation $y_n = c_1 3^n + c_2(1)^n$, we have

$$\left.\begin{aligned} y_{n+1} &= c_1 3^{n+1} + c_2(-1)^{n+1} = 3c_1 3^n - c_2(-1)^n \\ y_{n+2} &= c_1 3^{n+2} + c_2(-1)^{n+2} = 9c_1 3^n + c_2(-1)^n \end{aligned}\right\}$$

On eliminating c1 and c2 from above two relations, we get the desired difference equation of order 2 as

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & 3 & -1 \\ y_{n+2} & 9 & 1 \end{vmatrix} = 0 \quad \text{or} \quad \begin{aligned} y_{n+2}(-1-3) - y_{n+1}(1-9) + y_n(3+9) &= 0 \\ y_{n+2} - 2y_{n+1} - 3y_n &= 0 \end{aligned}$$

Example 5.2 : If $y_n = c_1 2^n + c_2 3^n + \frac{1}{2}$ find the corresponding difference equation.
Solution:

Given , $y_n = c_1 2^n + c_2 3^n + \frac{1}{2}$ or $\left(y_n - \frac{1}{2}\right) = c_1 2^n + c_2 3^n$ (1)
Implying $y_{n+1} = c_1 2^{n+1} + c_2 3^{n+1} + \frac{1}{2}$ or $\left(y_{n+1} - \frac{1}{2}\right) = 2c_1 2^n + 3c_2 3^n$ (2)
and $y_{n+2} = c_1 2^{n+2} + c_2 3^{n+2} + \frac{1}{2}$ or $\left(y_{n+2} - \frac{1}{2}\right) = 4c_1 2^n + 9c_2 3^n$ (3)

Elimination of c_1 and c_2 from (1), (2), (3) results in

$$\begin{vmatrix} y_n - \frac{1}{2} & 1 & 1 \\ y_{n+1} - \frac{1}{2} & 2 & 3 \\ y_{n+2} - \frac{1}{2} & 4 & 9 \end{vmatrix} = 0$$

Implying

$$\left(y_n - \frac{1}{2}\right)(18-12) - \left(y_{n+1} - \frac{1}{2}\right)(9-4) + \left(y_{n+2} - \frac{1}{2}\right)(3-2) = 0$$

$$6\left(y_n - \frac{1}{2}\right) - 5\left(y_{n+1} - \frac{1}{2}\right) + \left(y_{n+2} - \frac{1}{2}\right) = 0$$

Hence $y_{n+2} - 5y_{n+1} + 6y_n = 1$ the desired difference equation.

Example 5.3 : Form the difference equation from the relation given as

$$\frac{log(1+z)}{(1-z)} = y_0 + y_1 z + y_2 z^2 + - - - + y_n z^n$$

Solution: On rewriting the given relation as

$$log(1+z) = (1-z)(y_0 + y_1 z + y_2 z^2 + - - - + y_n z^n)$$

Implying

$$\left[z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + - - -\right] = (1-z)(y_0 + y_1 z + y_2 z^2 + - - - + y_n z^n)$$

$$\left[z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + - - -\right] = [y_0 + (y_1 - y_0)z + (y_2 - y_1)z^2 + (y_3 - y_2)z^3 + - - -]$$

On equating the coefficient of zn+1 on both sides

$$\left.\begin{aligned} (y_1 - y_0) &= \Delta y_0 = 1 \\ (y_2 - y_1) &= \Delta y_1 = -\tfrac{1}{2} \\ (y_3 - y_2) &= \Delta y_2 = \tfrac{1}{3} \\ (y_4 - y_3) &= \Delta y_3 = -\tfrac{1}{4} \\ &- - - - - - - - etc \end{aligned}\right\} \quad \text{for} \quad n=0,1,2,3,---$$

we get $y_{n+1} - y_n = \frac{(-1)^n}{n+1}$ i.e. $\Delta y_n = \frac{(-1)^n}{n+1}$ as the desired difference equation.

References

[1] Moy´e, L. A., Kapadia, A. S. (2000). Difference equations with public health applications. CRC Press.

[2] Potts, R. B. (1982). Differential and difference equations. The American Mathematical Monthly, 89(6), 402-407.

[3] Samuel, G. (1986). Introduction to Difference Equations: With Illustrative Examples from Economics, Psychology, and Sociology.

[4] Kelley, W. G., Peterson, A. C. (2001). Difference equations: an introduction with applications. Academic press.

[5] Sharkovsky, A. N., Maistrenko, Y. L. T., Romanenko, E. Y. (2012). Differ- ence equations and their applications (Vol. 250). Springer Science Business Media.

[6] Miller, K. (1960). The calculus of finite differences and difference equations.