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ROLE OF *(Α,Β)* METRIC IN HYPERSURFACE OF A FINSLER SPACE AND MTH ROOT METRICES IN RANDERS CHANGE OF FINSLER SPACE

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Abstract.

The purpose of present paper is devoted to studying a condition under which a Finsler space with Randers change of mth root is projectively related to a mth root metric and also to study hypersurfaces of special Finsler spaces.Further we have studied the conditions under which generalized Randers metric reduces to Rander metric and types of Finsler spaces arising from this metric and and also to investigate the various kinds of hypersurfaces of Finsler space with special (α,β) metric.

KEYWORDS: *Finsler space, Randers space, mth root metric, Hypersurfaces, Finsler spaces, C-reducible, quasi-C-reducible, p-reducible.*

1. Introduction

The study of spaces endowed with generalized metrics was initiated by P. Finsler in1918.The theory of hypersurfaces in general depends to a large extent on the study of the behavior of curves in them. The authors G.M. Brown, Moor, C. Shibata, M. Matsumoto, B.Y. Chen, C.S. Bagewadi, L.M. Abatangelo, Dragomir and S. Hojo have studied different properties of subspaces of Finsler, Kahler and Riemannian spaces. The concept of the (α,β)-metric $L(\alpha,\beta)$ was introduced by M. Matsumoto [17] and has been studied by many authors [7], [10], [25]. The study of some well known (α,β)-metrics, the Randers metric $\alpha+\beta$,the Kropina metric α^2/β and the generalized Kropina metric $\alpha^{(m+1)}/\beta^m$ have greatly contributed to the growth of Finsler geometry and its applications to theory of relativity.

A change of Finsler metric $F(x, y) \rightarrow \bar{F}(x, y)$ is called a generalized Randers change of F, if

$$\bar{F}(x, y) = F(x, y) + b_i(x, y)y^i$$

where$b_i(x, y)$ is h-vector in n-dimensional Finsler manifold (M,F). If a differential 1-form $\beta(x, y) =$

$b_i(x)y^i$ is given on M^n , **M,Matsumoto[10]** introduced another Finsler space whose fundamental function is given by

$$\bar{F}(x, y) = F(x, y) + \beta(x, y)$$ This

change of Finsler metric has been called β-change **[1,3].**

The mth root metric was firstly introduced by **Shimada [6]**in 1979.

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The second root metric is a Riemannian metric, so it is regarded as a generalization of Riemannian metric. Recent studies show that m^{th} rootFinsler metrics play a very important role in physics, space time and general relativity as well as in unified gauge field theory.**See ([4][14][12][15][5])**

Recently **S.H.Abed [11]**have generalized the metric with the help of h- vector and have introduced another Finsler metric defined as

$\bar{F}(x, y) = e^{\sigma(x)}F(x, y) + \beta(x, y)$

where$\sigma(x)$ is a function of x and $\beta(x, y) = b_i(x, y)y^i$ is a 1-form on M^n and b_i satisfies the condition of being an h-vector. We call the change $F(x, y) \to F(x, \_ y)$ as h-Randers conformal change. When $\beta = 0$, it reduces to a conformal change. When $\sigma = 0$, it reduces to a h- Randers change **[2].** When $\beta = 0$ and σ is a non-zero constant then it reduces to a homothetic change. It reduces to Randers conformal change when b_i *and* σ are functions of position only.**[11,8]**

2. Hypersurfaces of the Special Finsler Spaces

Now we consider the special Finsler spaces like Preducible, quasi-C-reducible, and C-reducible. Then we prove all these special Finsler space are well-defined in Finsler hypersurface F^{n-1} under some conditions.

Definition 2.1 (see **[17])** A Finsler space F^n is called a Preducible, if the torsion tensor P_{ijk} is written as

$$P_{ijk} = (h_{ij}P_k + h_{jk}P_i + h_{ki}P_j)/(n + 1) \qquad (2.1)$$

Where

$P_i = P^m_{im} = C_{i/0}$

Contracting (2.1) by $B_{\alpha\beta\gamma}{}^{ijk}$ and using $h_{\alpha\beta} = g_{\alpha\beta} - l_\alpha l_\beta$, and $h_{\alpha\beta} = h_{ij}B\alpha\beta ij$ we obtain

$PijkB\alpha\beta\gamma ijk = (hijPk + hjkPi + hkiPj)B\alpha\beta\gamma ijk /(n + 1),$

$PijkB\alpha\beta\gamma ijk = (h\alpha\beta P\gamma + h\beta\gamma P\alpha + h\gamma\alpha P\beta)/n,$

where we set. $P_i B^i_\alpha = P_\alpha = C_{\alpha/0}$ Hence we have the following result.

Theorem 2.2. A hypersurface of a P-reducible Finsler space is P-reducible.
Theorem 2.3 A hypersurface F^{n-1} of a quasi-C-reducible Finsler space F^n is quasi-C-reducible.

Suppose we assume that $C_\alpha = 0$, that implies

, $$C_iB^i_\alpha = 0 \qquad (2.2)$$
it means that C_i is tangential to the hypersurface F^{n-1}. then from (9), we have $C_{\alpha\beta\gamma} = 0$ therefore by Deickes
theorem the quasi-C-reducible Finsler hypersurface is Riemannian, which proves the following:

Theorem 2.4 A quasi-C-reducible Finsler hypersurface F^{n-1} is Riemannian, if the vector C_i is tangential to hypersurface F^{n-1}.

Definition 2.5 (see **[16])** A Finsler space $F^n(n>2)$ is said to be C-reducible, if it satisfies the equation

$$(n+1)C_{ijk} = h_{ij}C_k + h_{jk}C_i + h_{ki}C_j, \qquad (2.3)$$

where $C_i = g_{jk}C_{ijk}$

Contracting (2.3) by $B_{\alpha\beta\gamma}{}^{ijk}$ and using using the notations on Finsler hyper surface (see **[11], [18],[19]**): we obtain
$$nC_{\alpha\beta\gamma} = h_{\alpha\beta}C_\gamma + h_{\beta\gamma}C_\alpha + h_{\alpha\gamma}C_\beta, \qquad (2.4)$$ Where. $C_\alpha = C_iB^i_\alpha = g^{\beta\gamma}C_{\alpha\beta\gamma}$
Hence we have:

Theorem 2.6 *A hypersurface of a C-reducible Finsler space is C-reducible.*

Using the condition (2.2) in (2.4), we state that the following result:

Theorem 2.7 A hypersurface F^{n-1} of a C-reducible Finsler space is Riemannian, if the torsion vector C_i is tangential to hypersurface F^{n-1}.

3. The m-th root Metric

In 1941 ,**Randers [6]** introduced a Finsler metric $F = \alpha + \beta$ where $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannain metric and β
$= b_i(x)y^i$ is a differential one- form. In 1971 a Finsler metric

$$\bar{F}(x, y) = F(x, y) + \beta(x, y)$$
where F is a Finsler metric and β is a one-form on the manifold M^n , was introduced by M.Matsumoto. This
metric is called Randers change of Finsler metric.
Consider the transformation
$$\bar{F} = F + \beta \qquad (3.1)$$

Where $F = \sqrt[m]{A}$ is an m-th root metric and $\beta(x, y) = b_i(x)y^i$ is a one-form on the manifold M^n
.

The differentiation of (1.1) with respect to y^i yields the normalized supporting element $\bar{l_i}$ given by

$\bar{l_i} = l_i + b_i$

$l_i = \frac{Ai}{F^{m-1}} + b_{\bar{i}}$ as $l_i = \frac{Ai}{F^{m-1}}$ (3.2)

4. Randers change

In 1979, **Shimada[7]** introduced the m-th root metric on the differentiable manifold M defined as

$F = m\sqrt{ai1i2i3i4........im\ (x)yi1yi2yi3........yim}$

where the coefficients $a_{i_1 i_2 i_3 \ldots\ldots i_m}$ are the components of symmetric covariant tensor field of order (0,m)
being the functions of positional co-ordinates only.

There exist the following important two classes of Finsler metrics,

$$\bar{F} = \sqrt{A^{\frac{2}{m}} + B}$$

$$F = \sqrt{A^{\frac{2}{m}}} + B + \quad C \qquad \tilde{}$$

(4.1)
Where $A = a_{i_1 i_2 i_3 \ldots\ldots i_m}(x)y^{i_1}y^{i_2} \ldots .\ y^{i_m}$, $B = b_{ij}(x)y^i y^j$ and $C = c_k(x)y^k$, that is 1-form. These forms are called a generalized m-th root metric and more general generalized m-th root metric, respectively.
In [**14**], the geometric properties of locally projectively flat m-root in the form $F = \sqrt[m]{A}$ and generalized mth root in the form

$$\bar{F} = \sqrt{A^{\frac{2}{m}} + B}$$

Now we have considered a transformation of the more generalized m-th root metric such that it transforms to a similar metric as the generalized Randers

$\bar{F}(x, y) = F(x, y) + b_i(x, y)y^i$

In a way that the Finslerian metric F is replaced with more generalized m-th root metric $\tilde{F}$ defined in (2.1).

Then, we obtain the conditions among two more generalized m-th root metrics $\tilde{F}_1 = \sqrt{A_1^{\frac{2}{m_1}} + B_1 + C_1}$ and $\tilde{F}^2 = \sqrt{A_2^{\frac{2}{m_2}} + B_2 + C_2}$ due to generalized Randers change, when m_1, m_2 are even
numbers. We prove under these conditions generalized Randers metric reduces to Randers metric.

Theorem 4.1. *Let* $F_1 = \sqrt{A_1^{\frac{2}{m_1}} + B_1 + C_1}$ *and* $F^2 = \sqrt{A_2^{\frac{2}{m_2}} + B_2 + C_2}$ *are two more generalized m-th root metrics on an open subset* $U \subset R^n$. *Suppose that* m_1, m_2 *are even numbers with* $m_1 = m_2$ *and* $m_1, m_2 > 2$. *If* $\tilde{F}_1$ *is generalized Randers change of* $\tilde{F}_2$, *then* $\tilde{F}_1$ *reduces to a Randers* β*- change of* $\tilde{F}_2$.

Proof. Suppose that $\tilde{F}_1$ is generalized Randers change of $\tilde{F}_2$. Then

$C_1 = C_2 + b_i(x, y)y^i$ (4.2)

Differentiating (4.2) with respect to y^k we have

$C_k(x) = \bar{C_k}(x) + b_k(x, y)$ (4.3)

Then $\dot{\partial}_j b_k(x, y) = 0$ (4.4)

Therefore b_i are functions of coordinates x^i alone and b_i is not a h- vector.

5 .C-reducibility of $\bar{F^n}$

Following Matsumoto **[13]**, in this section we shall investigate special cases of the Finsler space with h- Randers conformally changes Finsler space $\bar{F}^n$.

Definition 5.1 A Finsler space (M^n, L) with dimension $n \geq 3$ is said to be quasi-C-reducible if the Cartan tensor C_{ijk} satisfies

$Cijk = QijCk + QjkCi + QkiCj$

Where Q_{ij} is a symmetric indicatory tensor.

We have **[8]**

$$\bar{C}^h_{ij} = C^h_{ij} + \frac{1}{2\bar{L}}\left(h_{ij}m^h + h^h_j m_i + h^h_i m_j\right) - \frac{1}{L}\left[\left\{\rho + \frac{L}{2\bar{L}}(b^2 - \frac{\beta^2}{L^2}\right\}h_{ij} + \frac{L}{\bar{L}}m_i m_j\right]l^h$$

Substituting $h = j$ in (3.1) we get

$$C_i = C_i + \frac{n+1}{2\bar{L}}m_i \quad (5.1)$$

Using **[12]** and (5.1) we have

$$Cijk = \varphi Cijk + \frac{\varphi}{(n+1)} \pi(ijk)\{hij(Ck - Ck)\}$$

Where $\pi_{(ijk)}$ represents cyclic permutation and sum over the indices i, j and k The above equation can be written as

$$\bar{C}_{ijk} = \varphi C_{ijk} + \frac{\varphi}{(n+1)} \pi_{(ijk)}\left(h_{ij}\bar{C}_k\right) - \frac{\varphi}{(n+1)} \pi_{(ijk)}\left(h_{ij}C_k\right)$$

Definition 5.2 A Finsler space (M^n, L) of dimension $n \geq 3$ is called C-reducible if the Cartan tensor C_{ijk} is written in the form

$$C_{ijk} = \frac{1}{(n+1)} (h_{ij}C_k + h_{ki}C_j + h_{jk}C_i) \tag{5.2}$$

Now from **[12]** and definition of C-reducibility we have

$$\varphi C_{ijk} = \pi_{ijk()}(\bar{h}_{ij} N_k) \tag{5.3}$$

Where $N_k = \frac{1}{(n+1)} \bar{C}_k - \frac{1}{2\tau} m_k$

Conversely, if (5.3) is satisfied or certain vector N_k then we have

$$\bar{Cijk} = \frac{1}{(n+1)} \pi(ijk)(\bar{h}ij\bar{Ck})$$

Thus we have

Theorem 5.3. *An h-Randers conformally changed Finsler Space $\bar{F^n}$ is C-reducible iff equation (3.4) holds good.*

Corollary 5.1. If the Finsler space F^n is C-reducible Finsler space, then an h-Randers conformally changed Finsler space $\bar{F^n}$ is always a C-reducible Finsler space.

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