UTILISING A NOVEL APPROACH TO THE FRACTIONAL CAUCHY-TYPE EQUATIONS

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Abstract:
The fractional of Cauchy-type problem (FCT) is solved precisely in the current study utilizing the Laplace residual power series approach (LRPS). The Caputo operator is used to determine the fractional derivative. Firstly, we present a brand-new technique that combines the residual power series strategy with the Laplace transform technique. We provide precise instructions for employing the suggested methodology to calculate fractional Cauchy-type formula. Next, we assess the technique's effectiveness and accuracy using the FCT. The calculated and actual results are examined using graphic representations of the results, demonstrating how much more accurate the proposed approach is. The table illustrates the findings for fractional approximations results for different fractional orders in addition to nonfractional approximations and correct results. It is shown that like the number of phrases inside the serial that solve the issues rises, the relation between the generated answers as well as the real solutions to every issue converges. To exemplify that how proposed scheme works in calculating various types of fractional ordinary differential equations, two instances are provided.
Keywords: Fractional power series, Fractional Cauchy-type formula, Series of Laplace residual power, Laurent's series.

Introduction

Fractional derivatives (FD), which generalize integer derivatives and shift the rank of derivatives from integer to real or indeed complexes, are useful in helping explain a variety of phenomena. Numerous applications in a range of scientific domains have led to the development of a distinctive mathematical strategy for solving problems known as fractional calculus (FC) [1,2,3]. FC is effectively used in many areas, like biology, processing of images and signals, finance, and physics. The origin of FC is briefly addressed in every one of the numerous publications on the subject which have recently come out [4,5,6,7]. The core subjects of FC include consistent non - linear model, abnormal dispersion, controlling, and vibrations [8]. According to Podliubny [6], having reviewed several implementations that have evolved

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from numerical techniques of viscoelasticity, it comes so easily to apply fractional calculus. It was possible to consider many fractional derivative types, including Caputo, Hadamard, Grünwald-Letnikov and Riemann-Liouville. Diverse approaches have been put forth to describe fractional differential equations, including technique of Yang transform decomposition [9,10], technique of auxiliary formula [11], technique of fractional variational iteration [12,13], technique of collocation [14], technique of trapezoidal [15], technique of homotopy analysis [16], technique of Elzaki transform decomposition [17,18], technique of homotopy perturbation transform [19,20] and numerous others [21,22,23,24].

In order to calculate the parameters for the power series solutions for the fractions and non-fractional DEs, it is possible to use the RPS [25], [26], [27], [28], [29], [30]. It relies on building power series answers to a variety of nonlinear and linear problems and offers the answer in the form of a convergent series not linearization, and disruption, or separation. The fractional KDV-Burgers formula, fractional Schrödinger formulas, fractional multi-pantograph structure, and several other kinds of a fraction ordinary DEs, a fraction partial DEs, all have been resolved effectively by the RPS technique. Because it is outdated, certain types of DEs are still solved using the Laplace transform technique.

An ordinary differential equation in second rank describes the Cauchy-type formula [31]. At the same time, we provide this as

$$D_{\eta}^{\alpha}u(\eta)-\xi u(\eta)=\mathrm{z}(\eta),\ \ \eta>0,0<\alpha\leq 1, \tag{1}$$

And initial condition as:

$$D-+$$

$\alpha-m\|*u(\eta)=\lambda_m, m=1,2,\ldots,n$

here $\mathrm{z}(\eta)$ is a predetermined function.

This manuscript's primary goal is to examine analytical and approximative answers to Cauchy-type problems using the Laplace residual power series (LRPS) method, which was suggested and demonstrated in [36]. The LRPS approach combines the Laplace transform approach and the RPS approach, providing both precise and approximative results as quickly as fractional power series (FPS) answers as a consequence of transforming the main issue to Laplace space and developing answers to novel algebraic problems. The main issue is then resolved by applying the Laplace inverse of the outcomes. In contrast to the FRPS method, that relies on the fractional derivative and can require longer to compute the various fractional derivatives in stages in order to identify the results, the unidentified parameters in the novel Laplace expansions may be identified by employing the limit idea. The LRPS method has fewer time- and accuracy-intensive minor computing demands.

The significance of this study is in compared the correct result of non-FCT to fifth-rank approximations for a variety of fractional derivative values with the exact result of an FCT utilizing a comparable new approach. This work may serve as the primary guide by researchers to determine this approach and apply it in a variety of contexts to obtain precise and approximate findings in some few simple steps. Our study's application of LRPSM for FCT in simple, straightforward procedures is one of its distinctive features. We give clear foundations and features of fractional derivatives in Part 2. The recommended strategy is presented in Part 3, and two FCT situations are accurately solved in Part 4.

Preliminaries

In this part, we also cover the Laplace transform results and the fundamental idea of fractional calculus:

Definition 1 [32]. The fractional derivative is the same in the Caputo as:

$$^{C}D^{\alpha}w(\eta) = \mathrm{J}^{\delta-\alpha}w^{\delta}(\eta),\ \delta-1<\alpha\leq\delta$$

(2)

where the Riemann-Liouville (RL) integral operators is represented by J^{α} as

*

$$\mathrm{J}^{\alpha}w(\eta) = \frac{1}{\Gamma(\alpha)}\int_0^{\kappa}\ (\kappa-t)^{\alpha-1}w(t)dt$$

(3)

and $\delta \in \mathbb{N}$

Definition 2. [33] The Laplace transform (LT) defined on function $w(\eta)$ is

$$\mathcal{L}\{w(\eta)\} = \int_0^{\infty}\ e^{-s\eta}w(\eta)d\eta,\ s>\alpha$$

(4)

making use of inverse LT as follows:

$$\mathcal{L}^{-1}\{W(s)\} = \int_{c-i\infty}^{c+i\infty}\ e^{s\eta}W(s)ds,\ c=Re(s)>c_0$$

(5)

Lemma 3. [34] If we assume that $w(\eta)$ is a piece - wise continuous function with $W(s) = \mathcal{L}\{w(\eta)\}$ the below characteristics are genuine:

(i) $\mathcal{L}\{\mathrm{J}_*^{\alpha}w(\eta)\} = \frac{W(s)}{s^{\alpha}},\ \varrho>0$

(ii) $\mathcal{L}\{\mathrm{D}_*^{\alpha}w(\eta)\} = s^{\alpha}W(s) - \sum_{i=0}^{k-1}s^{\alpha-k-1}w^{k}(0),\ k-1<\alpha\leq k;$

(iii) $\mathcal{L}\{\mathrm{D}_*^{k\alpha}w(\eta)\} = s^{k\alpha}W(s) - \sum_{i=0}^{k-1}s^{(k-i)\alpha-1}\mathrm{D}_*^{i\alpha}w(0),\ 0<\alpha\leq 1.$

Proposition 4. [33] Take into account that $w(\eta)$ is a piecewise continuous on $[0,\infty)$ with an exponential order of $\mathfrak{I}$. Consider that the fractional expansions of $W(s) = \mathcal{L}\{w(\eta)\}$ will be as follows:

$$W(s) = \sum_{m=0}^{\infty}\ \frac{\lambda_m}{s^{1+m\alpha}},\ 0<\alpha\leq 1, s>\mathfrak{I}$$

(6)

Hence, $\lambda_m = \mathrm{D}_*^{i\alpha}w(0)$.

Remark 5.[33] Using inverse LT to the provided (6), we get:

$$w(\eta) = \sum_{m=0}^{\infty} \frac{\mathbf{D}_{*}^{\alpha} w(0)}{\Gamma(1+m\alpha)} \eta^{m\alpha},\ 0 < \alpha \leq 1, \eta \geq 0$$

(7)

It is comparable to the fractional Taylor's equation presented in [33].

Definition 6. [35] An expanding that represents the below:

$$\sum_{m=-\infty}^{\infty} \lambda_m (t - t_0)^m = \sum_{m=1}^{\infty} \frac{\lambda_{-m}}{(\eta-\eta_0)^m} + \sum_{m=0}^{\infty} \lambda_m (\eta - \eta_0)^m,$$

(8)

is referred to as a (LS) at (LS), where η is a parameter and λ_n's are constant known as series' factors. The analytical or regular section of Laurent's series seems to be the series $\sum_{m=-\infty}^{\infty} \lambda_m (\eta - \eta_0)^m$. Although the single or main section of Laurent's series is defined by $\sum_{m=1}^{\infty} \frac{\lambda_{-m}}{(\eta-\eta_0)^m}$.

The Cauchy-Type Equations Solutions Using LRPS Technique

Take the following fractional Cauchy-Type differential to demonstrate how the LRPS technique may be used to create a series solution to the FODEs:

$$D_{\eta}^{\alpha} u(\eta) - \xi u(\eta) = \mathrm{z}(\eta),\ \ \eta > 0, 0 < \alpha \leq 1,$$

(9)

And initial condition as:

$$D_{\eta}^{\alpha-m} u(\eta) = \lambda_m, m = 1,2, \dots, n$$

(10)

In the beginning, use the LT to (9), we obtain:

$$\mathcal{L}\left[D_{\eta}^{\alpha} u(\eta) - \xi u(\eta)\right] = \mathcal{L}[\mathrm{z}(\eta)],\ \ \eta \in\ I.$$

(11)

with I is an open interval and $\mathrm{z}(t)$ is an analytical function.

We may construct (11) as following using Lemma 3:

$$s^{\alpha} U(s) - s^{\alpha-1}\, u(0) - \xi U(s) = Z(s),\ s > 0.$$

(12)

where $U(s) = \mathcal{L}[u(\eta)]$ and $Z(s) = \mathcal{L}[z(\eta)]$.

The next form of Eq. (12) is produced by dividing it by s^{α} and applying the beginning circumstances from Eq. (12):

$$U(s) = \frac{\lambda_0}{\mathrm{s}} + \frac{\xi}{s^{\alpha}} U(s) + \frac{Z(s)}{s^{\alpha}},\ s > 0.$$

(13)

Consider that extension of Eq. (13) result is as follows:

$$\boldsymbol{U(s)} = \sum_{\boldsymbol{j=0}}^{\infty} \frac{\boldsymbol{\lambda_j}}{\boldsymbol{s^{1+\alpha j}}}\ , \boldsymbol{s > 0}.$$

(14)

According to (14), the kth-truncated series is

$$U_k(s) = \frac{\lambda_0}{\mathrm{s}} + \sum_{j=1}^{k} \frac{\lambda_j}{s^{1+\alpha j}}\ , s > 0.$$

(15)

We can define the main LRPS techniques like the LRF of Eq. (13), in order to determine the unknown value of the parameter, λ_j that is presented as:

$$\mathrm{LRes(s)} = \mathrm{U}(s) - \frac{\lambda_0}{\mathrm{s}} - \frac{\xi}{s^{\alpha}} U(s) - \frac{Z(s)}{s^{\alpha}}, s > 0.$$

(16)

thus, the kth-LRF defined as:

$$\mathrm{LRes}_k(\mathrm{s}) = \mathrm{U}_k(s) - \frac{\lambda_0}{\mathrm{s}} - \frac{\xi}{s^{\alpha}} \mathrm{U}_k(s) - \frac{Z(s)}{s^{\alpha}}, s > 0.$$

(17)

It is obvious that for $s > 0$ and $k = 0,1,2,3, ...$ $\mathrm{Lim}_{k\to\infty}\mathrm{LRes}_k(s) = \mathrm{LRes}(s)$, $\mathrm{LRes}(s) = 0$. As a result, $\mathrm{Lim}_{s\to\infty}\left(s^k \mathrm{LRes}(s)\right) = 0$. Additionally, it was established [36, 37] and

$$\mathrm{Lim}_{s\to\infty}\left(s^{k+1}\mathrm{LRes}(s)\right) = \mathrm{Lim}_{s\to\infty}\left(s^{k+1}\mathrm{LRes}_k(s)\right) = 0, k = 1,2,3, ..$$

(18)

Given that $\mathrm{U}_1(s) = \frac{\lambda_0}{\mathrm{s}} + \frac{\boldsymbol{\lambda_1}}{\boldsymbol{s^{1+\alpha}}}$, Eq. (17) signify:

$$\mathrm{LRes}_1(\mathrm{s}) = \frac{\boldsymbol{\lambda_1}}{\boldsymbol{s^{1+\alpha}}} - \frac{\xi\boldsymbol{\lambda_0}}{s^{\alpha+1}} - \frac{\xi\boldsymbol{\lambda_1}}{\boldsymbol{s^{1+2\alpha}}} - \frac{Z(s)}{s^{\alpha}}, s > 0.$$

(19)

Next, multiplying $\boldsymbol{s^{1+\alpha}}$ by two parts of equation (19) yields

$$\boldsymbol{s^{1+\alpha}}\mathrm{LRes}_1(\mathrm{s}) = \boldsymbol{\lambda_1} - \xi\boldsymbol{\lambda_0} - \frac{\xi\boldsymbol{\lambda_1}}{\boldsymbol{s^{\alpha}}} - sZ(s), s > 0.$$

(20)

Next, utilizing the assumption in Eq. (18) and the limit as $s \to \infty$ from both parts of Eq. (20), we may quickly ascertain the value of λ_1 via resolving the formula given of λ_1:

$$\begin{aligned} 0 &= \lambda_1 - \xi\lambda_0 - \mathrm{Lim}_{s\to\infty} sZ(s) \\ &= \lambda_1 - \xi\lambda_0 - z(0), s > 0. \end{aligned}$$

(21)

It is simple to get the following by calculating λ_1 in the ensuing algebraic formula (21).

$$\lambda_1 = \xi\lambda_0 + z(0), s > 0.$$

(22)

The 2nd-truncated series of Eq. (17), $\mathrm{U}_2(s) = \frac{\lambda_0}{\mathrm{s}} + \frac{\lambda_1}{s^{1+\alpha}} + \frac{\lambda_2}{s^{1+2\alpha}}$, is substituted into in the 2nd -LRF to calculate the value of the next undetermined parameter λ_2 as follows:

$$\mathrm{LRes}_2(\mathrm{s}) = \frac{\lambda_2}{s^{1+2\alpha}} - \frac{\xi\lambda_1}{s^{2\alpha+1}} - \frac{\xi\lambda_2}{s^{1+3\alpha}} - \frac{Z(s)}{s^{\alpha}}, s > 0.$$

(23)

Next, multiplying $s^{1+2\alpha}$ by two parts of equation (23) yields

$$s^{1+2\alpha}\mathrm{LRes}_2(\mathrm{s}) = \lambda_2 - \xi\lambda_1 - \frac{\xi\lambda_2}{s^{\alpha}} - s^{1+\alpha}Z(s), s > 0.$$

(24)

To get the below formula, calculate the limit as $s \to \infty$ with both parts of Eq. (24) and then employ Eq. (18).

$$\begin{aligned} 0 &= \lambda_2 - \xi\lambda_1 - \mathrm{Lim}_{s\to\infty} s^{1+\alpha}Z(s) \\ &= \lambda_2 - \xi\lambda_1 - \left(D_\eta^{\alpha} z(0)\right), s > 0. \end{aligned}$$

(25)

By resolving the algebraic equation that results for λ_2, we obtain

$$\lambda_2 = \xi\lambda_1 + \left(D_\eta^{\alpha} z(0)\right), s > 0.$$

(26)

Similar to a previous stages, replace the 3rd -truncated series of Eq. (17), $\mathrm{U}_3(s) = \frac{\lambda_0}{\mathrm{s}} + \frac{\boldsymbol{\lambda_1}}{\boldsymbol{s^{1+\alpha}}} + \frac{\boldsymbol{\lambda_2}}{\boldsymbol{s^{1+2\alpha}}} + \frac{\lambda_3}{\boldsymbol{s^{1+3\alpha}}}$ is substituted into in the 3rd -LRF to calculate the value of the next undetermined parameter $\boldsymbol{\lambda_3}$ as follows:

$$\mathrm{LRes}_3(\mathrm{s}) = \frac{\boldsymbol{\lambda_3}}{\boldsymbol{s^{1+3\alpha}}} - \frac{\xi\boldsymbol{\lambda_2}}{s^{3\alpha+1}} - \frac{\xi\boldsymbol{\lambda_3}}{\boldsymbol{s^{1+4\alpha}}} - \frac{Z(s)}{s^{\alpha}}, s > 0.$$

(27)

multiplying $\boldsymbol{s^{1+3\alpha}}$ by two parts of equation (27) yields

$$s^{1+3\alpha}\mathrm{LRes}_3(\mathrm{s}) = \lambda_3 - \xi\lambda_2 - \frac{\xi\lambda_3}{s^{\alpha}} - s^{1+2\alpha}Z(s), s > 0.$$

(28)

To get the below formula, calculate the limit as $s \to \infty$ with both parts of Eq. (28) and then employ Eq. (18).

Using the fact (18) and the limit as $s \to \infty$ both for parts of Eq. (28), we arrive as:

$$\begin{aligned} 0 &= \lambda_3 - \xi\lambda_2 - \mathrm{Lim}_{s\to\infty} s^{1+2\alpha}Z(s) \\ &= \lambda_3 - \xi\lambda_2 - \left(D_\eta^{2\alpha} z(0)\right), s > 0. \end{aligned}$$

(29)

For λ_3, resolving equation (29) yields

$$\lambda_3 = \xi\lambda_2 + \left(D_\eta^{2\alpha} z(0)\right), s > 0.$$

(30)

We can readily determine the factor λ_k by looking at the pattern of the derived factors, which is as continues to follow:

$$\lambda_k = \xi\lambda_{k-1} + \left(D_\eta^{(k-1)\alpha} z(0)\right), s > 0, k = 1,2,\ldots.$$

(31)

As a result, we may write the Eq. (15) result's in an infinite series like described in the following:

$$U(s)=\frac{\lambda_0}{s}+\sum_{j=1}^{\infty}\left(\frac{\lambda_j}{s^{1+\alpha j}}+D_\eta^{(k-1)\alpha}z(0)\right).$$

(32)

The LRPS solution to Equations (9) and (10) is obtained by using the inverse LT of Equation (32) in the given simple format:

$$u(\eta)=\lambda_0+\sum_{j=1}^{\infty}\left(\frac{\lambda_j}{s^{1+\alpha j}}+D_\eta^{(k-1)\alpha}z(0)\right)\frac{\eta^{\alpha j}}{\Gamma(1+\alpha j)}.$$

(33)

Results and discussion

In this part, we look at the importance of the LRPSM in obtaining the solution to the CT.

Problem 1: Take into account the fractional equation below:

$$D_\eta^\alpha u(\eta)-5u(\eta)=0\,,\quad \eta>0, 0<\alpha\leq 1,$$

(34)

And initial condition as:

$$u(0)=1$$

(35)

Using (35), the LT is taken to (34) which gives us

$$U(s)=\frac{1}{s}+\frac{5}{s^\alpha}U(s),\ s>0.$$

(36)

It is claimed that the kth-truncated series is

$$U_k(s)=\frac{1}{s}+\sum_{j=1}^{k}\frac{\lambda_j}{s^{1+\alpha j}}\ ,s>0.$$

(37)

Consequently, the kth LRFs are

$$\mathrm{LRes}_k(\mathrm{s})=\mathrm{U}_k(s)-\frac{1}{s}-\frac{5}{s^\alpha}\mathrm{U}_k(s), s>0.$$

(38)

The kth-truncated series (37) is now placed into the kth LRF (38) to give λ_j. After multiplying the resultant formula by $s^{1+\alpha j}$, we may calculate the relationship.

$$\mathrm{Lim}_{s\to\infty}\left(s^{k+1}\mathrm{LRes}_k(s)\right)=0, k=1,2,3,..$$

So, several values include:

$$\lambda_1=1,$$
$$\lambda_2=1,$$
$$\lambda_3=1,$$
$$\lambda_4=1,$$
$$\lambda_5=1,$$

And so on.

We next obtain the values of λ_j via entering them in (37), where $j=1,2,3,\ldots$

$$U(s) = \frac{1}{s} + \frac{1}{s^{1+\alpha}} + \frac{1}{s^{1+2\alpha}} + \frac{1}{s^{1+3\alpha}} + \frac{1}{s^{1+4\alpha}} + \ldots$$

(39)

If we calculate LT's inverse, we obtain

$$u(\eta) = 1 + \frac{\eta^{\alpha}}{\Gamma(1+\alpha)} + \frac{\eta^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{\eta^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{\eta^{4\alpha}}{\Gamma(1+4\alpha)} + \frac{\eta^{5\alpha}}{\Gamma(1+5\alpha)} + \ldots$$

(40)

If $\alpha = 1$, then Eq. (40) become as follows:

$$u(\eta) = 1 + \eta + \frac{\eta^2}{2} + \frac{\eta^3}{6} + \frac{\eta^4}{24} + \frac{\eta^5}{120} + \ldots$$

(41)

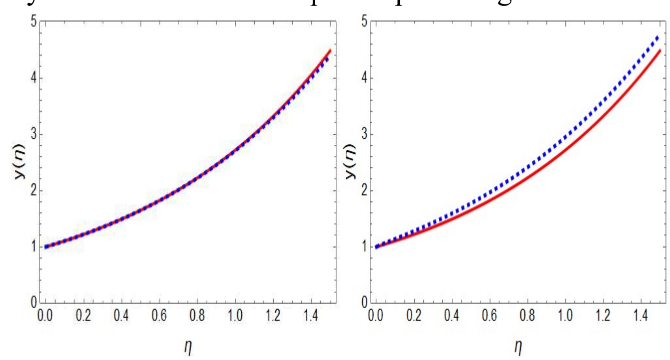
The exact solution of Eqs. (34), (35) is:

$$u(\eta) = e^{\eta}$$

(42)

The graphs of a 5th approximation and exact result to Eqs. (34) and (35) in the range [0,5] is shown in Figure 4.1. The graphic shows that there is a large area in which the approximation result and the exact result match.

The absolute error, together with the accurate and approximation findings at various values of t inside the interval [0,1], are provided in Table 4.1a and b because the exact result of the Eqs. (34) and (35) are known to exist. According to the outcomes, the LRPS approach is a reliable analytical numerical technique for providing exact results to the FODEs.



(a) (b)

Figure 1. The actual result of Eq. (34), (35) and the 5th approximation of the LRPS solution. The full curve shows
the exact result, while the dotted curve the approximation LRPS result at (a) $\alpha = 1$, (b) $\alpha = 0.90$.

Table 1.a The actual error and the 5th approximative LRPS result for Eqs. (34), (35) at $\alpha = 1$

η	$u_5(\eta)$ −Approximation	$u(\eta)$- Exact	$Act.Err.(\eta)$
0.0	1	1	0
0.1	1.10517	1.10517	8.47423×10^{-8}
0.2	1.2214	1.2214	2.75816×10^{-6}
0.3	1.34984	1.34986	2.13076×10^{-5}
0.4	1.49173	1.49182	9.13643×10^{-5}
0.5	1.64844	1.64872	2.83771×10^{-4}
0.6	1.8214	1.82212	7.188×10^{-4}
0.7	2.01217	2.01375	1.58187×10^{-3}
0.8	2.2224	2.22554	3.14093×10^{-3}
0.9	2.45384	2.4596	5.76561×10^{-3}

Table 1.b The actual error and the 5th approximative LRPS result for Eqs. (34), (35) at $\alpha = 0.90$

η	$u_5(\eta)$ −Approximation	$u(\eta)$- Exact	$Act.Err.(\eta)$
0.0	1	1	0
0.1	1.14085	1.10517	35.6772×10^{-3}
0.2	1.28052	1.2214	59.116×10^{-3}
0.3	1.43041	1.34986	80.5462×10^{-3}
0.4	1.5934	1.49182	10.1579×10^{-2}
0.5	1.77155	1.64872	12.2826×10^{-2}
0.6	1.96663	1.82212	14.4511×10^{-2}
0.7	2.18037	2.01375	16.6616×10^{-2}
0.8	2.41448	2.22554	18.8935×10^{-2}
0.9	2.67068	2.4596	21.1076×10^{-2}

Problem 4.2: Take into account the fractional equation below:

$$D_{\eta}^{\alpha} u(\eta) - u(\eta) = e^{x}, \quad \eta > 0, 0 < \alpha \leq 1, \tag{43}$$

And initial condition as:

$$u(0) = 0$$

(44)

Using (44), the LT is taken to (43) which gives us

$$U(s) = \frac{1}{s^{\alpha}}U(s) + \frac{e^{\eta}}{s^{\alpha}},\ s > 0.$$

(45)

It is claimed that the kth-truncated series is

$$U_k(s) = \sum_{j=1}^{k} \frac{\lambda_j}{s^{1+\alpha j}}\ , s > 0.$$

(46)

Consequently, the kth LRFs are

$$\mathrm{LRes}_k(s) = \mathrm{U}_k(s) - \frac{1}{s^{\alpha}}\mathrm{U}_k(s) + \frac{e^{x}}{s^{\alpha}}, s > 0.$$

(47)

The kth-truncated series (46) is now placed into the kth LRF (47) to give $\boldsymbol{\lambda_j}$. After multiplying the resultant formula by $\boldsymbol{s^{1+\alpha j}}$, we may calculate the relationship.

$$\mathrm{Lim}_{s\to\infty}\left(s^{k+1}\mathrm{LRes}_k(s)\right) = 0, k = 1,2,3, ..$$

So, several values include:

$$\lambda_1 = -e^{\eta},$$
$$\lambda_2 = 2e^{\eta},$$
$$\lambda_3 = 3e^{\eta},$$
$$\lambda_4 = 4e^{\eta},$$
$$\lambda_5 = 5e^{\eta},$$

And so on.

We next obtain the values of $\boldsymbol{\lambda_j}$ via entering them in (46), where $j = 1,2,3, \ldots$

$$U(s) = \frac{-e^{\eta}}{\boldsymbol{s^{1+\alpha}}} + \frac{2e^{\eta}}{\boldsymbol{s^{1+2\alpha}}} + \frac{3e^{\eta}}{\boldsymbol{s^{1+3\alpha}}} + \frac{4e^{\eta}}{\boldsymbol{s^{1+4\alpha}}} + \frac{5e^{\eta}}{\boldsymbol{s^{1+5\alpha}}} + \ldots$$

(48)

If we calculate LT's inverse, we obtain

$$u(\eta) = -\frac{e^{\eta}\eta^{\alpha}}{\Gamma(1+\alpha)} + \frac{2e^{\eta}\eta^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{3e^{\eta}\eta^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{4e^{\eta}\eta^{4\alpha}}{\Gamma(1+4\alpha)} + \frac{5e^{\eta}\eta^{5\alpha}}{\Gamma(1+5\alpha)} + \ldots$$

(49)

If $\alpha = 1$, then Eq. (49) become as follows:

$$u(\eta) = -e^{\eta}\eta + \frac{2e^{\eta}\eta^2}{2} + \frac{3e^{\eta}\eta^3}{6} + \frac{4e^{\eta}\eta^{4\alpha}}{24} + \frac{5e^{\eta}\eta^5}{120} + \ldots$$

(50)

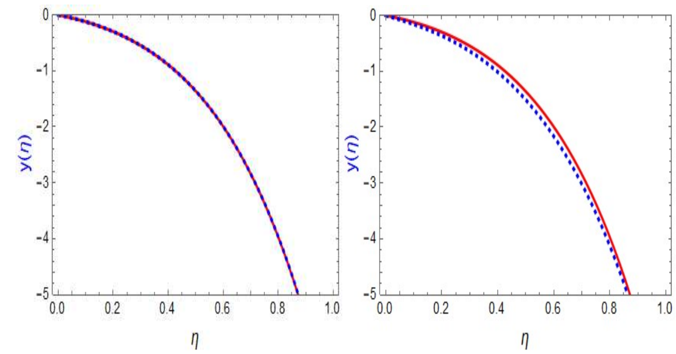
The exact solution of Eqs. (43), (44) is:

$$u(\eta) = -\eta e^{2\eta}$$

(51)

The graphs of a 5th approximation and exact result to Eqs. (43) and (44) in the range [0,5] is shown in Figure 4.2. The graphic shows that there is a large area in which the approximation result and the exact result match.

The absolute error, together with the accurate and approximation findings at various values of t inside the interval [0,1], are provided in Table 4.1 a and b because the exact result of the Eqs. (43) and (44) are known to exist. According to the outcomes, the LRPS approach is a reliable analytical numerical technique for providing exact results to the FODEs.



(a) (b)

Figure 2. The actual result of Eq. (43), (44) and the 5th approximation of the LRPS solution. The full curve shows the exact result, while the dotted curve the approximation LRPS result at (a) $\alpha = 1$, (b) $\alpha = 0.90$.

Table 2.a The actual error and the 5th approximative LRPS result for Eqs. (43), (44) at $\alpha = 1$

η	$u_5(\eta)$	$u(\eta)$	$Act.Err.(\eta)$
0.0	0	0	0
0.1	-0.010202	-0.010202	9.36547×10^{-9}
0.2	-0.298364	-0.298365	6.73765×10^{-7}
0.3	-0.546627	-0.546636	8.62867×10^{-6}
0.4	-0.890162	-0.890216	5.45198×10^{-5}
0.5	-1.35891	-1.35914	2.33929×10^{-4}
0.6	-1.99128	-1.99207	7.85844×10^{-4}
0.7	-2.83641	-2.83864	2.22985×10^{-3}
0.8	-3.95683	-3.96243	5.59221×10^{-3}
0.9	-5.43192	-5.44468	12.763×10^{-3}

Table 2.b The actual error and the 5th approximative LRPS result for Eqs. (43), (44) at $\alpha = 0.90$

η	$u_5(\eta)$	$u(\eta)$	$Act.Err.(\eta)$
0.0	1	1	0
0.1	1.14085	1.10517	35.6586×10^{-3}
0.2	1.28052	1.2214	64.5542×10^{-3}
0.3	1.43041	1.34986	94.0477×10^{-3}
0.4	1.5934	1.49182	12.3505×10^{-2}
0.5	1.77155	1.64872	15.0419×10^{-2}
0.6	1.96663	1.82212	17.0303×10^{-2}
0.7	2.18037	2.01375	17.603×10^{-2}
0.8	2.41448	2.22554	15.6807×10^{-2}
0.9	2.67068	2.4596	9.67511×10^{-2}

Conclusion

This work presents a novel approach using the Laplace transform and residual power series to address numerous important linear fractional Cauchy-Type problems. The benefit of the new approach is that it involves less calculation to arrive at the answer in series arrangement, where parameters are determined through a sequence of algebraic procedures. The proposed scheme was used to answer two different prototype, and tables and graphs demonstrated its accuracy. Lastly, we showed that fractional problems may be handled with great precision and straightforward computations using the Laplace residual power series technique. The acquired findings have been displayed in tables and figures. We found that the exact and analytical solutions are strongly related to from the graphs and tables. The tables and graphs helped us to understand how nearly the accurate and analytical results are linked to each other. Smaller computations made with the current technique have higher precision and may be applied in a later review to extend the LRPS techniques to bidimensional practical situations. The recommended method can also be used to examine a variety of fractional issues connected to the transmission of linear events in physics research.

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