TENSOR PRODUCT OF REPRESENTATION FOR LIE GROUPS

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Abstract:
Our main interest in this work is to study the properties and basics of the Lie group. Through what we offer from double representations of the Lie group, and we linked the tensor product and the double representations to obtain new characteristics. In this paper we use the quadrilateral Z action (QZA). We researched the relationship between (QZA) of Lie group and tensor product, we will put in this relationship new structures consisting of five vector spaces by double representations. At the end of the research, we showed the relationship between the Lie group and the Lie algebra through the tables that were clarified.
These proposals are developed and substantiated through observations and sources.

Keywords: Lie group, tensor product, tensor product of Lie group , representation of Lie group.

1. Introduction:

Lie group Ǥ is a smooth manifold which being as well the group, such that a multiplication function m: Q × Q, (g, h) → gh as well as a attaching of an i: Q → Q, g → g^{-1} be smooth inverse maps[6].Lie algebra is a vector space over some field to gather with bilinear multiplications is called the bracked which satisfies :Bilinearity , Jacobi identity and Anti – symmetric [3] . By Schur's lemma, new starting point review of Lie algebra on the space of linear maps from W_2 into W_1 , Hom (W_2 ,W_1), and Hom (W_2 , W_1) ≅ $W^*_2 \otimes W_1$,[1] and [8]. The representation for Lie group has been addressed ,and the tensor product, which are explained in the equivalence relationship between them using a representations that contains five vector spaces with the use of a quadrilateral Z action (QZA) ,and to clarify all aspects related to the changes this new structures. And we will show the relationship between (QZA) of Lie groups and (QZA) Lie algebras using the exponential map .See [10] and [4] .

2. The equivalence between (QZA) of Lie group and tensor product.

Definition (2.1): [5]

A representation for Lie group Q is a finithed. The dimensional real (complex) vector space W together with the Q → Q(W) is homomorphism of Lie group . It means r: Q → Q(W) isthe depiction of Lie group Q where r is linear map and dim $W \geq 1$.

Definition (2.2): [9]

Let $Q_1$and Q_2 be two Lie groups and let r_1 and r_2 be representations of Q_1 and Q_2 affects the spaces W_1 and W_2 respectively. Where the tensor product of r_1 and r_2 signified by $r_1 \otimes r_2$ is a representation of $Q_1 \otimes Q_2$ conducting on $W_1 \otimes W_2$ denoted via $r_1 \otimes r_2(y, z) = r_1(y) \otimes r_2(z)$, for all $y \in Q_1$ and $z \in$.Q_2

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Example (2.3): [9]

Let $r_1 = S^1 \to (2, ₵)$ be representation of S^1 where
$S^1 = \{(c\,\vartheta, sin\ \vartheta\ ), 0 \le \vartheta \le 2\pi\}, S^1 = e^{iP} = \cos\vartheta + i\ \sin\vartheta$ and

$S0(2, ₵) = \left\{\begin{pmatrix}\cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta\end{pmatrix}, 0 \le \vartheta \le 2\pi\right\}$ such that

$$r_1(y) = r_1(e^{i\vartheta}) = r_1(\cos\vartheta, \sin\vartheta) = \begin{pmatrix}\cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta\end{pmatrix}, 0 \le \vartheta \le 2\pi$$

And let $r_2 = \mathbb{R}^2 \to (3)$ be representation of $\mathbb{R}^2$ such that

$r_2(z) = r_2(e^{i\vartheta}) = \begin{bmatrix}1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0\end{bmatrix}$ then

$$(r_1 \otimes r_2)(y, z) = r_1(y) \otimes r_2(z) = r_1(e^{i\vartheta}) \otimes r_2(e^{i\vartheta})$$

$$= \begin{pmatrix}\cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta\end{pmatrix} \otimes \begin{bmatrix}1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0\end{bmatrix}\ (r_1 \otimes r_2)(y, z)$$

$$= \begin{bmatrix} \cos\vartheta \begin{bmatrix}1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0\end{bmatrix} & -\sin\vartheta \begin{bmatrix}1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0\end{bmatrix} \\ \sin\vartheta \begin{bmatrix}1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0\end{bmatrix} & \cos\vartheta \begin{bmatrix}1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0\end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\vartheta & 0 & 0 & -\sin\vartheta & 0 & 0 \\ 0 & 0 & -\cos\vartheta & 0 & 0 & \sin\vartheta \\ 0 & \cos\vartheta & 0 & 0 & -\sin\vartheta & 0 \\ \sin\vartheta & 0 & 0 & \cos\vartheta & 0 & 0 \\ 0 & 0 & -\sin\vartheta & 0 & 0 & \cos\vartheta \\ 0 & \sin\vartheta & 0 & 0 & \cos\vartheta & 0 \end{bmatrix}_{6\times6}$$

Definition (2.4): [2]

Let r be a representation for Lie group Q affects the finite dimensional vector space W, then the double depiction r^* *to* r is the depiction of Q conducting on W^* provided via : $r^*(y) = (r(y)^{-1})^{tr}$, and the double depiction being also name contragredient .

Example (2.5):[2]

Let $r = S^1 \to So(2, ₵)$, where $S^1 = e^{i\vartheta} = \cos\vartheta + i\ \sin\vartheta$ such that

$$r(\cos\vartheta\ , \sin\vartheta) = \begin{pmatrix}\cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta\end{pmatrix}\ , r(e^{i\vartheta}) = \begin{pmatrix}\cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta\end{pmatrix},$$

r is representation of Lie group S^1. Then

$$g = e^{i\vartheta}\ , r(e^{i\vartheta})$$

$$= \begin{pmatrix}\cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta\end{pmatrix},$$

$$g^{-1} = \cos\vartheta - i\ \sin\vartheta$$

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Proposition (2.6):

Let r_{i} , $i = 1,2,3,4,5$ be the representations of Lie group Q affects M-finite dimensionality spaces W_{i},
$i = 1,2,3,4,5$, where M be field, and then the (QZA) for Lie group of Q on:

$$Hom_M\left(Hom\left(W_5^*, Hom(W_4, W_3^*) \oplus Hom(W_3^*, W_2)\right)\right), W_1)$$
$$(r_5 \otimes \left((r_4^* \otimes r_3) \oplus (r_3 \otimes r_2^*)\right)) \otimes r_1^* \text{ of Q on :}$$
$$GL(W_5^* \otimes (W_4^*, W_3) \oplus (W_3, W_2^*) \otimes W_1^*).$$

Proof:

For revealing that,

$$Э: (W_5^* \otimes (W_4^*, W_3) \oplus (W_3, W_2^*) \otimes W_1^*) \to Hom_M\,((W_5^*, (W_4, W_3^*) \oplus (W_3^*, W_2))), W_1).$$

Is a bilinear map, define by:
$Э(w_5, w_1^*) = K$, for all $w_5 \in W_5$ and $w_1^* \in W_1$,
where $K: W_5 \to W_1$ is a linear map defines by:

$$K(c) = w_5(c)w_1^*, \text{ for all } \acute{w}_5, w_5 \in W_5, c \in W_5, A, B \in M, w_1^* \in W_1 .$$
$$K(Aw_5 + B\acute{w}_5, w_1^*) = (Aw_5 + B\acute{w}_5(c))w_1^*$$
$$= Aw_5(c)w_1^* + B\acute{w}_5(c)w_1^*$$
$$= AЭ(w_5, w_1^*) + BЭ(\acute{w}_5, w_1^*)$$

The other for the whole is $w_1^*, \acute{w}_1^* \in W_1^*$ and $w_5 \in W_5$

$$Э(w_5, Aw_1^* + B\acute{w}_1^*) = W_5(c)(Aw_1^* + B\acute{w}_1^*)$$
$$= w_5(c)(Aw_1^*) + (w_5(c)(B\acute{w}_1^*)$$
$$= Aw_5(c)w_1^* + Bw_5(c)\acute{w}_1^*$$

And on it,

$$Э(W_5 \times ((W_4^* \times W_3) \oplus (W_3, W_2^*))) \times W_1^*$$
$$\to Hom_M\left(Hom\left(W_5^*, Hom(W_4, W_3^*) \oplus Hom(W_3^*, W_2)\right)\right), W_1)$$

Is the bilinear map, therefore via utilizing the tensor product as well as the universal property of this tensor product, we get a unique linear map Ω.

$$(W_5 \times ((W_4^* \times W_3) \oplus (W_3, W_2^*) \longrightarrow ((W_5 \times ((W_4^* \times W_3) \otimes (W_3, W_2^*))) \otimes W_1^*)$$

Bilinear map Э ↘   Ω ↙ linear map

$$\left(Hom\left(W_5^*, Hom(W_4, W_3^*) \oplus Hom(W_3^*, W_2)\right)\right), W_1$$

So , via the general characteristics of the tensor product,

$$((W_5 \times ((W_4^* \times W_3) \oplus (W_3, W_2^*))) \times W_1^*)$$

And, there's a distinctive linear map:

$$Ω: ((W_5 \times ((W_4^* \times W_3) \oplus (W_3, W_2^*))) \times W_1^*)$$
$$\to \left(Hom\left(W_5^*, Hom(W_4, W_3^*) \oplus Hom(W_3^*, W_2)\right)\right), W_1)$$

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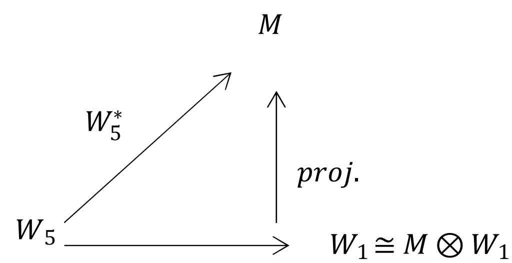
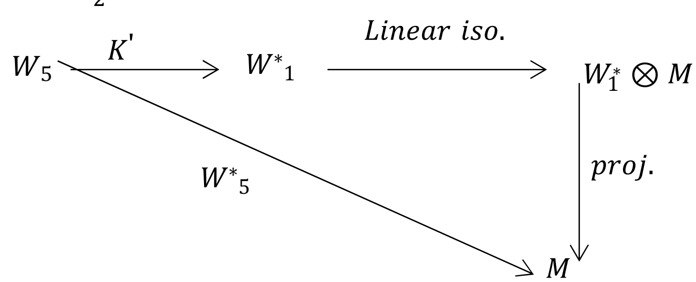
And on it , (via universal property of tensor product, there exists a distinctive linear map) , this creates the overhead diagram as a commutative



Consider the linear maps composition, where $W_5(c)$ being defined as the following:
$K(c) = W_1^*$. $\exists!\, m \in M$, such that $W_1^* \to (m, W_1^*)$, because the whole maps being linear , m is distinctive, and putting $W_5(c) = m$ associated with W_1^* is to define:
$\varphi: Hom\big(W_5^*, Hom(W_4, W_3^*) \oplus Hom(W_3^*, W_2)\big), W_1 \to ((W_5 \times ((W_4^* \times W_3) \oplus (W_3, W_2^*))) \times W_1^*)$
distinctive, and putting $W_5(c) = m$ associated with W^* is to define:
φ: $(W^*, (W_4, W^*) \oplus Ho(W^*, W_2)), W_1$

By
Define : $\varphi(\acute{K}) = W_5(c)W_1^*$,
$W_5 \to W_5(c)W_1^*$ by $W_5(c) = m$, where m is given by:
$\varphi\acute{K}(c) = (m, \acute{K}(c))$, we can show that W_5 is linear map. $\acute{K}(c) = W_1^*$, for all
$\acute{K} \in Hom_M\Big(Hom\big(W_5^*, Hom(W_4, W_3^*) \oplus Hom(W_3^*, W_2)\big)\Big), W_1)$, $w_1^* \in W_1^*, w_5 \in W_5$ and is related to W_1^*
$\acute{K}(Aa_1 + Ba_2) = A\acute{K}(a_1) + B\acute{K}(a_2) = Aw_1^* + Bw_2^* = Aw_5(c) + Bw_5(c)$, for all $w_5 \in W_5$.
Where $W_5(c) = m_1 \Longrightarrow Aw_5(c_1), w_5(Ac_1 + Bc_2) = Am_1 + Bm_2$.
$W_5(c_2) = m_2 \Longrightarrow w_5(Ac_2) = Bm_2$.



Clear K' is linear and $\varphi^{-1} = \Omega^{-1}$, therefor (φ) is linear map. Related the (QZA) of the Liegroup of Q on:
$Hom_M\Big(Hom\big(W_5^*, Hom(W_4, W_3^*) \oplus Hom(W_3^*, W_2)\big)\Big), W_1)$ and the (QZA) of the Lie group of Q on:
$(W_5 \otimes ((W_4^* \otimes W_3) \oplus (W_3 \otimes W_2^*))) \otimes W_1^*)$, up to the representation given:

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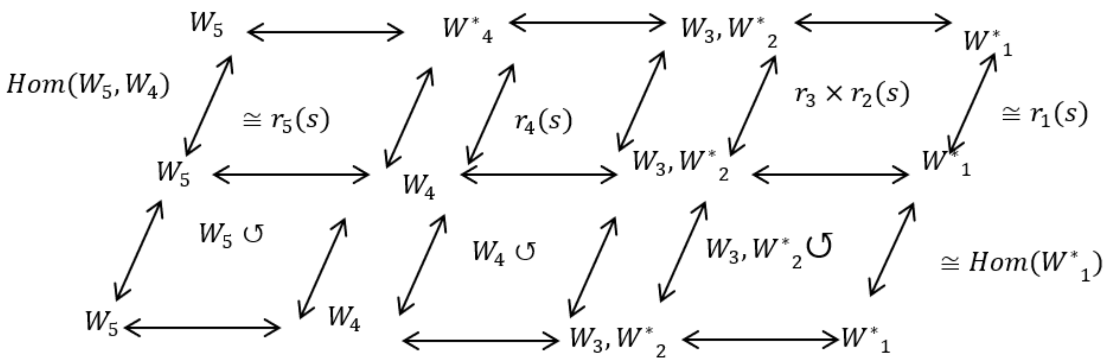


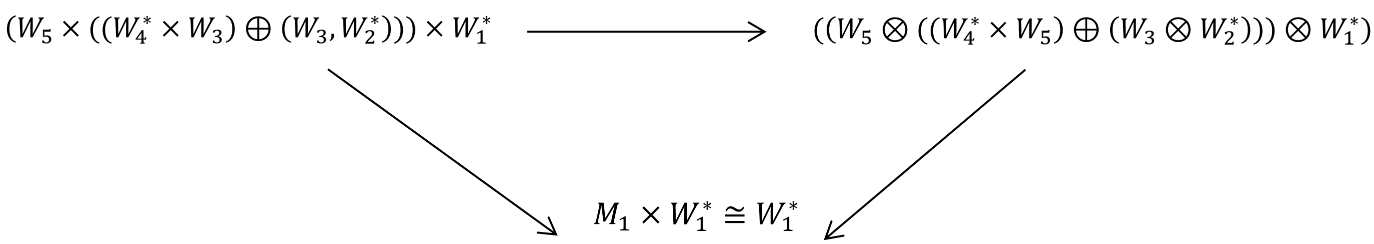
Notation (2.7): [2]

Generally, let n is a natural number ,so it will be:

n is the number of even , $W_n^{\overbrace{(*,*\ldots)\ldots*}} \cong W$,

n is the number of odd , $W_n^{\overbrace{(*,*\ldots)\ldots*}} \cong W^*$,

Define φ via $\varphi(w_5, w_1^*) = w_5(c)w_1^*$, for all $w_5 \in W_5$ and $K(c) = W_1^*$.



And because of the universal property it exists a unique Э define via:
$Э(W_5 \otimes ((W_4^* \otimes W_3) \oplus (W_3 \otimes W_2^*))) \otimes W_1^* = w_5(c)K(w_5^*)$ for all $w_5 \in W_5$, $w_1^* = K(w_5^*) \in W_1$ and $K \in Hom_M\left(Hom\left(W_5^*, Hom(W_4, W_3^*) \oplus Hom(W_3^*, W_2)\right)\right), W_1)$

Proposition (2.8):

Suppose that $r_1, r_2, r_3, r_4 and\ r_5$ are matrix representation for (QZA) of Q affects on W_1, W_2, W_3, W_4, W_5 respectively, then (QZA) of the Lie group on:
$(W_5 \otimes ((W_4^* \otimes W_3) \oplus (W_3 \otimes W_2^*))) \otimes W_1^*)$ is satisfied:

$$(W_5 \otimes ((W_4^* \otimes W_3) \oplus (W_3 \otimes W_2^*))) \otimes W_1^*)$$
$$\cong Hom_M\left(Hom\left(W_5^*, Hom(W_4, W_3^*) \oplus Hom(W_3^*, W_2)\right)\right), W_1)$$

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Proof:

With existence two maps, invertible linear map , intertwining map for the action of Q on:
$(W_5 \otimes ((W_4^* \otimes W_3) \oplus (W_3 \otimes W_2^*))) \otimes W_1^*)$ into
$Hom_M\left(Hom\left(W_5^*, Hom(W_4, W_3^*) \oplus Hom(W_3^*, W_2)\right)\right), W_1)$ such that:

$\ni(W_5 \otimes ((W_4^* \otimes W_3) \oplus (W_3 \otimes W_2^*))) \otimes W_1^* \to Hom_M\left(Hom\left(W_5^*, Hom(W_4, W_3^*) \oplus Hom(W_3^*, W_2)\right)\right), W_1)$.

$\ni\left(r_5 \otimes \left((r_4^* \otimes r_3) \oplus (r_3 \otimes r_2^*)\right)\right) \otimes r_1^*(r) = (\left(r_5 K_4 r_4\left((K_3 r_4 K_3 r_3) \oplus (r_3 K_2 r_2)\right) K_1 r_1\right)(r) = r(r) K_4 r_4(r) K_3 r_3(r) \oplus r_3(r) K_2 r_2(r) K_1 r_1(r)$.

Step1/ by notation(2.7) we have
$(W_5 \otimes ((W_4^* \otimes W_3) \oplus (W_3 \otimes W_2^*)) \otimes W_1) \cong Hom(W_5, \widetilde{W})$ such that
$\widetilde{W} = ((W_4^* \otimes W_3) \oplus (W_3 \otimes W_2^*) \otimes W_1)$, let $W_5, \widetilde{W} = W$.

Step2/ by notation(2.7) we have $W \otimes W_1^* \cong Hom(W, W_1^*)$.
By step (1) and step (2):
$(W_5 \otimes ((W_4^* \otimes W_3) \oplus (W_3 \otimes W_2^*)) \otimes W_1) \cong Hom_M\left(Hom\left(W_5^*, Hom(W_4, W_3^*) \oplus Hom(W_3^*, W_2)\right)\right), W_1$.

Proposition (2.9):

Let r_i, $i = 1,2,3,4,5$ be matrix representation of W_i, $i = 1,2,3,4$ *and* 5 respectively ,then (QZA) for Lie group Q on
$Hom_M\left(Hom\left(W_5^*, Hom(W_4, W_3^*) \oplus Hom(W_3^*, W_2)\right)\right), W_1^*)$ the depiction of r, such that:
$r^*(y) = [r_5(y) K_4(r_4(y^{-1}) \oplus r_3(y) K_2(r_2(y^{-1}))] K_1 r_1(y)$. For all $y \in$ Q, then the (QZA) of Lie group on:
$\left[Hom_M\left(Hom\left(W_5^*, Hom(W_4, W_3^*) \oplus Hom(W_3^*, W_2)\right)\right), W_1^*)\right]^*$
Is also given by dual representation r^*, such that
$r^*(y) = r_1^*(y) K_1^*(r_2^*(y^{-1}) K_2^* r_2^*(y)) \oplus (r_3^*(y^{-1}) K_3^* r_2^*(y) K_4^* r_5^*(y^{-1})$.

Proof:

The (QZA) for Lie group Q on
$Hom_M\left(Hom\left(W_5^*, Hom(W_4, W_3^*) \oplus Hom(W_3^*, W_2)\right)\right), W_1^*)$
Is induced by the representation:
$r(y) = [r_5(y^{-1}) K_4(r_4(y) K_3 r_3(y) \oplus (r_3(y) K_2 r_2(y^{-1}))] K_1 r_1(y)$. For all $y \in$ Q.
$K_1 \in W_1$, $K_2 \in (W_3 \times W_2)$, $K_3 Hom_m(W_1, W_2 \times W_3) K_4 \in Hom_m(W_5 \times W_4)$.
To show that:
$r^*(y)$: Q $\to QL\left(Hom_M\left(Hom\left(W_5^*, Hom(W_4, W_3^*) \oplus Hom(W_3^*, W_2)\right)\right), W_1^*)\right)^*$.
Is a representation, such that:
$r^*(y) = r_1^*(y) K_1^*(r_2^*(y^{-1}) K_2^* r_2^*(y)) \oplus (r_3^*(y^{-1}) K_3^* r_2^*(y) K_4^* r_5^*(y^{-1})$ for all $y \in$ Q and
$K_4^* \in \left(Hom(W_5 \times W_4)^*, K_3^* \in \left(Hom(W_1, W_2 \times W_3)\right), K_2^* \in Hom(W_3 \times W_2)^*, K_1^* \in W_1\right)^*$.
Since $r^*(y) = (r_5(y^{-1}) K_4 \oplus r_4(y) K_3 r_3(y^{-1})) \oplus \left(r_3(y) K_2 r_2(y^{-1}) K_1 r_1(y)\right)$

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$r^*(y) = r_1^*(y)K_1^*(r_2^*(y^{-1})K_2^*r_2^*(y)) \oplus (r_3^*(y^{-1})K_3^*r_2^*(y)K_4^*r_5^*(s^{-1})K_4^*r_5(y^{-1}))$ for all $y \in$ Q .
And $K_4^*: W_5^* \to W_4^*$ and $r^*(yz) = (r(yz))^* = (r(y)r(z))^* = r^*(y)r^*(z)$
For all y , $z \in$ Q .
And on it, r^* is a representation from Q on

$$\left(Hom_M\left(Hom\left(W_5^*, Hom(W_4, W_3^*) \oplus Hom(W_3^*, W_2)\right)\right), W_1^*)\right)^*$$

Proposition (2.10):

Let $W_i, i = 1,2,3,4,5$ are five vector spaces, W_i^* is the dual of vectors W_i, $i = 1,2,3,4,5$ then the following assertions being equivalent to:

1. $\left(Hom_M\left(Hom\left(W_5^*, Hom(W_4, W_3^*) \oplus Hom(W_3^*, W_2)\right)\right), W_1^*)\right)^*$.
2. $Hom_M\left(W_1^*, Hom(W^{**}_2, W_3^*) \oplus Hom(W^{**}_3, W_4^*), HomW^{**}_5)\right)$.
3. $Hom_M(W_1^*, Hom(W_2, W_3^*) \oplus Hom(W_3, W_4^*), Hom(W_5, M))$.
4. $Hom_M(W_1^*, Hom(W_2, W_3^*) \oplus Hom(W_3(W_4^*, M), Hom(W_5))$.
5. $Hom_M(W_1^*, Hom(W_2, M), W_3^*) \oplus Hom(W_3, W_4^*), Hom(W_5))$.
6. $Hom_M(W_1^*, M)Hom(W_2, W_3^*) \oplus Hom(W_3, W_4^*), Hom(W_5))$.
7. $\left(Hom_M\left(Hom\left(W_5^*, Hom(W_4, W_3^*) \oplus Hom(W_3^*, W_2)\right)\right), W_1)\right)^{\overbrace{*,*,\dots,*}^{n}}$

$$= \begin{cases} \left(Hom_M\left(Hom\left(W_5^*, Hom(W_4, W_3^*) \oplus Hom(W_3^*, W_2)\right)\right), W_1)\right) \text{ if } n \text{ is an even number} \\ \left(Hom_M\left(Hom\left(W_5^*, Hom(W_4, W_3^*) \oplus Hom(W_3^*, W_2)\right)\right), W_1^*)\right) \text{ if } n \text{ is an odd number} \end{cases}$$

Proof:

(1) ≅ (2) to show: $\left(Hom_M\left(Hom\left(W_5^*, Hom(W_4, W_3^*) \oplus Hom(W_3^*, W_2)\right)\right), W_1)\right)^*$
$\cong Hom_M\left(W_1^*, Hom(W^{**}_2, W_3^*) \oplus Hom(W^{**}_3, W_4^*), HomW^{**}_5)\right)$
Let: $K_4 \in Hom_M(W_5 \times W_4), K_3 \in Hom_M(W_1, W_2 \times W_3), K_2 \in (W_3 \times W_2), K_1 \in W_1$
Such that:
$K_4^* \in Hom_M(W_4{}^* \times W_5{}^*), K_3^* \in Hom_M\left((W_3{}^* \times W_2{}^*), W_1^*\right)K_2^* \in (W_2{}^* \times W_3{}^*), K_1^* \in W_1^*$
And, there exists an intertwining map:
$\ni Hom_M(Hom\left(W_5^*, Hom(W_4, W_3^*) \oplus Hom(W_2, W_3^*)\right), W_1^*) \to Hom_M(Hom(Hom(W_4, W_3)^* W_1^*), W_5^*))$
Such that
$\ni\left(r^*(y)\right)(r) = r^*(y)\ni r$, for all $r \in W_1^*$.
Also, if we take 6 instead of 7, we get an equivalent .
By the same methods, we prove the remaining parts.

Example (2.11):

Consider that, $r_1 = S^1 \to So(2) \subseteq QL(2, M) \cong QL(W_1)$ such that
$r_1(e^{i\vartheta}) = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}, 0 \le \vartheta \le 2\pi.$
$r_2 = S^1 \to So(2) \subseteq QL(2, M) \cong QL(W_2)$ such that
$r_2(e^{i\vartheta}) = \begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}, 0 \le \vartheta \le 2\pi.$

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$r_3 = S^1 \to o(3) \subseteq \mathrm{Q}L(23, M) \cong \mathrm{Q}L(W_3)$ such that

$r_3(e^{i\vartheta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & -\sin\vartheta \\ 0 & \sin\vartheta & \cos\vartheta \end{pmatrix}, 0 \le \vartheta \le 2\pi$

$r_4 = S^1 \to \mathrm{Q}L(2, M) \cong \mathrm{Q}L(W_4)$, such that

$r_4(e^{i\vartheta}) = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}, 0 \le \vartheta \le 2\pi.$

$r_5 = S^1 \to \mathrm{Q}L(2, M) \cong \mathrm{Q}L(W_5)$, such that

$r_5(e^{i\vartheta}) = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix}, 0 \le \vartheta \le 2\pi.$

Then (QZA) for the Lie group Q on:

$\left(Hom_M\left(Hom(W_5^*, Hom(W_4, W_3^*) \oplus Hom(W_3^*, W_2))\right), W_1\right)$ is a representation:

$$r^*(y) = r_1^*(y)K_1^*(r_2(y^{-1})K_2^*r_3(y)) \oplus (r_3^*(y^{-1})K_3^*r_4(y)K_4^*r_5^*(y^{-1}))$$

$$= r^*(e^{i\vartheta}) = r_1^*(e^{i\vartheta})K_1^*(r_2(e^{i\vartheta})^{tr}K_2^*r_3(e^{i\vartheta})) \oplus (r_3^*(e^{i\vartheta}) \oplus r_3(e^{-i\vartheta})^{tr}K_3^*r_4(e^{i\vartheta})K_4^*r_5^*(e^{-i\vartheta})^{tr}$$

$$= (\begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} K_1 \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} K_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & -\sin\vartheta \\ 0 & \sin\vartheta & \cos\vartheta \end{pmatrix})$$

$$\oplus (\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & -\sin\vartheta \\ 0 & \sin\vartheta & \cos\vartheta \end{pmatrix} K_3 \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} K_4 \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix})$$

$$= \begin{pmatrix} \cos^2\vartheta & \cos\vartheta\sin\vartheta & \sin\vartheta\cos\vartheta & \sin^2\vartheta \\ -\cos\vartheta\sin\vartheta & \cos^2\vartheta & \sin^2\vartheta & \sin\vartheta\cos\vartheta \\ -\sin\vartheta\cos\vartheta & \sin^2\vartheta & \cos^2\vartheta & \cos\vartheta\sin\vartheta \\ \sin^2\vartheta & -\sin\vartheta\cos\vartheta - & \cos\vartheta\sin\vartheta & \cos^2\vartheta \end{pmatrix}_{4\times4} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & -\sin\vartheta \\ 0 & \sin\vartheta & \cos\vartheta \end{pmatrix}_{3\times3}$$

$$\begin{pmatrix} \begin{matrix} \cos^2\vartheta & \cos\vartheta\sin\vartheta & \sin\vartheta\cos\vartheta & \sin^2\vartheta \\ -\cos\vartheta\sin\vartheta & \cos^2\vartheta & \sin^2\vartheta & \sin\vartheta\cos\vartheta \\ -\sin\vartheta\cos\vartheta & \sin^2\vartheta & \cos^2\vartheta & \cos\vartheta\sin\vartheta \\ \sin^2\vartheta & -\sin\vartheta\cos\vartheta - & \cos\vartheta\sin\vartheta & \cos^2\vartheta \end{matrix} & 0 & 0 \\ 0 & \begin{matrix} \cos^3\vartheta & \cos\vartheta\sin\vartheta & \sin\vartheta\cos\vartheta & \sin^2\vartheta \\ -\cos\vartheta\sin\vartheta & \cos^3\vartheta & \sin^2\vartheta & \sin\vartheta\cos\vartheta \\ -\sin\vartheta\cos\vartheta & \sin^2\vartheta & \cos^2\vartheta & \cos\vartheta\sin\vartheta \\ \sin^2\vartheta & -\sin\vartheta\cos\vartheta - & \cos\vartheta\sin\vartheta & \cos^2\vartheta \end{matrix} & \begin{matrix} \cos^2\vartheta & \cos\vartheta\sin\vartheta & \sin\vartheta\cos\vartheta & \sin^3\vartheta \\ -\cos\vartheta\sin\vartheta & \cos^2\vartheta & \sin^3\vartheta & \sin\vartheta\cos\vartheta \\ -\sin\vartheta\cos\vartheta & \sin^2\vartheta & \cos^2\vartheta & \cos\vartheta\sin\vartheta \\ \sin^3\vartheta & -\sin\vartheta\cos\vartheta - & \cos\vartheta\sin\vartheta & \cos^2\vartheta \end{matrix} \\ 0 & \begin{matrix} \cos^2\vartheta\sin\vartheta & \cos\vartheta\sin\vartheta & \sin\vartheta\cos\vartheta & \sin^2\vartheta \\ \cos\vartheta\sin^2\vartheta & \cos^2\vartheta\sin\vartheta & \sin^2\vartheta & \sin\vartheta\cos\vartheta \\ -\sin^2\vartheta\cos\vartheta & \sin^3\vartheta & \cos^2\vartheta & \cos\vartheta\sin\vartheta \\ \sin^3\vartheta & -\sin^2\vartheta\cos\vartheta - & \cos\vartheta\sin\vartheta & \cos^3\vartheta \end{matrix} & \begin{matrix} \cos^3\vartheta & \cos\vartheta\sin\vartheta & \sin\vartheta\cos\vartheta & \sin^3\vartheta \\ -\cos\vartheta\sin\vartheta & \cos^3\vartheta & \sin^2\vartheta & \sin\vartheta\cos\vartheta \\ -\sin\vartheta\cos\vartheta & \sin^2\vartheta & \cos^3\vartheta & \cos\vartheta\sin\vartheta \\ \sin^2\vartheta & -\sin\vartheta\cos\vartheta - & \cos\vartheta\sin\vartheta & \cos^2\vartheta \end{matrix} \end{pmatrix}_{12\times12}$$

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$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & -\sin\vartheta \\ 0 & \sin\vartheta & \cos\vartheta \end{pmatrix}_{3\times 3} \otimes \begin{pmatrix} \cos^2\vartheta & \cos\vartheta\sin\vartheta & \sin\vartheta\cos\vartheta & \sin^2\vartheta \\ -\cos\vartheta\sin\vartheta & \cos^2\vartheta & \sin^2\vartheta & \sin\vartheta\cos\vartheta \\ -\sin\vartheta\cos\vartheta & \sin^2\vartheta & \cos^2\vartheta & \cos\vartheta\sin\vartheta \\ \sin^2\vartheta & -\sin\vartheta\cos\vartheta - & \cos\sigma\sin\vartheta & \cos^2\vartheta \end{pmatrix}_{4\times 4} =$$

$$\begin{pmatrix}
\begin{matrix} \cos^2\vartheta & \cos\vartheta\sin\vartheta & \sin\vartheta\cos\vartheta & \sin^2\vartheta \\ -\cos\vartheta\sin\vartheta & \cos^2\vartheta & \sin^2\vartheta & \sin\vartheta\cos\vartheta \\ -\sin\vartheta\cos\vartheta & \sin^2\vartheta & \cos^2\vartheta & \cos\vartheta\sin\vartheta \\ \sin^2\vartheta & -\sin\vartheta\cos\vartheta - & \cos\vartheta\sin\vartheta & \cos^2\vartheta \end{matrix} & 0 & 0 \\
0 & \begin{matrix} \cos^3\vartheta & \cos^2\vartheta\sin\vartheta & \sin\vartheta\cos^2\vartheta & \sin^2\vartheta\cos\vartheta \\ \cos^2\vartheta\sin\vartheta & \cos^3\vartheta & \sin^2\vartheta\cos\vartheta & \sin\vartheta\cos^2\vartheta \\ -\sin\vartheta\cos^2\vartheta & -\sin^2\vartheta\cos\vartheta & \cos^3\vartheta & \cos^2\vartheta\sin\vartheta \\ \sin^2\vartheta\cos\vartheta & -\sin\vartheta\cos\vartheta & \cos^2\vartheta\sin\vartheta & \cos^3\vartheta \end{matrix} & \begin{matrix} \sin\vartheta\cos^2\vartheta & -\cos\vartheta\sin^2\vartheta & \sin^2\vartheta\cos\vartheta & \sin^3\vartheta \\ -\cos\vartheta\sin^2\vartheta & \cos^2\vartheta\sin\vartheta & \sin^3\vartheta & \sin^2\vartheta\cos\vartheta \\ -\sin\vartheta\cos\vartheta & \sin^2\vartheta & \cos^2\vartheta & \cos\vartheta\sin\vartheta \\ -\sin^3\vartheta & \sin^2\vartheta & \cos\vartheta - \cos^2\vartheta \quad \sin\vartheta & \cos^2\vartheta\sin\vartheta \end{matrix} \\
0 & \begin{matrix} \sin\vartheta\cos^2\vartheta & \cos\vartheta\sin^2\vartheta & \sin^2\vartheta\cos\vartheta & \sin^3\vartheta \\ \cos\vartheta\sin^2\vartheta & \cos^2\vartheta\sin\vartheta & \sin^3\vartheta & \sin^2\vartheta\cos\vartheta \\ -\sin^2\vartheta\cos\vartheta & \sin^3\vartheta & \cos^2\vartheta\sin\vartheta & \cos\vartheta\sin\vartheta \\ \sin^3\vartheta & -\sin^2\vartheta\cos\vartheta & -\cos\vartheta\sin^2\vartheta & \cos^2\vartheta\sin\vartheta \end{matrix} & \begin{matrix} \cos^3\vartheta & \cos^2\vartheta\sin\vartheta & \sin^2\vartheta\cos\vartheta & \sin^2\vartheta\cos\vartheta \\ \cos^2\vartheta\sin\vartheta & \cos^3\vartheta & \sin^2\vartheta & \sin\vartheta\cos\vartheta \\ -\sin\vartheta\cos\vartheta & \sin^2\vartheta & \cos^3\vartheta & \cos\vartheta\sin\vartheta \\ \sin^2\vartheta\cos\vartheta & -\sin\vartheta\cos^2\vartheta & \cos^2\vartheta\sin\vartheta & \cos^3\vartheta \end{matrix}
\end{pmatrix}_{12\times 12}$$

$$\begin{pmatrix}
\begin{matrix} 2\cos^2\vartheta & 2\cos\vartheta\sin\vartheta & 2\sin\vartheta\cos\vartheta & 2\sin^2\vartheta \\ -2\cos\vartheta\sin\vartheta & 2\cos^2\vartheta & 2\sin^2\vartheta & 2\sin\vartheta\cos\vartheta \\ -2\sin\vartheta\cos\vartheta & 2\sin^2\vartheta & 2\cos^2\vartheta & 2\cos\vartheta\sin\vartheta \\ 2\sin^2\vartheta & -2\sin\vartheta\cos\vartheta - & 2\cos\vartheta\sin\vartheta & 2\cos^2\vartheta \end{matrix} & 0 & 0 \\
0 & \begin{matrix} 2\cos^3\vartheta & 2\cos^2\vartheta\sin\vartheta & 2\sin\vartheta\cos^2\vartheta & 2\sin^2\vartheta\cos\vartheta \\ 2\cos^2\vartheta\sin\vartheta & 2\cos^3\vartheta & 2\sin^2\vartheta\cos\vartheta & 2\sin\vartheta\cos^2\vartheta \\ -2\sin\vartheta\cos^2\vartheta & -2\sin^2\sigma\cos\sigma & 2\cos^3\vartheta & 2\cos^2\vartheta\sin\vartheta \\ \sin^2\vartheta\cos\vartheta & -\sin\sigma\cos\sigma & \cos^2\vartheta\sin\vartheta & \cos^3\vartheta \end{matrix} & \begin{matrix} -2\sin\vartheta\cos^2\vartheta & -2\cos\vartheta\sin^2\vartheta & -2\sin^2\vartheta\cos\vartheta & -2\sin^3\vartheta \\ -2\cos\vartheta\sin^2\vartheta & -2\cos^2\vartheta\sin\vartheta & -2\sin^3\vartheta & -2\sin^2\vartheta\cos\vartheta \\ 2\sin\vartheta\cos\vartheta & 2\sin^2\vartheta & -2\cos^2\vartheta & -2\cos\vartheta\sin\vartheta \\ -2\sin^3\vartheta & 2\sin^2\vartheta\cos\vartheta & 2\cos^2\vartheta\, 2\sin\vartheta & 2\cos^2\vartheta\sin\vartheta \end{matrix} \\
0 & \begin{matrix} 2\cos^2\vartheta\sin\vartheta & 2\cos\vartheta\sin^2\vartheta & 2\sin^2\vartheta\cos\vartheta & 2\sin^3\vartheta \\ 2\cos\vartheta\sin^2\vartheta & 2\cos^2\vartheta\sin\vartheta & 2\sin^3\vartheta & 2\sin^2\vartheta\cos\vartheta \\ -2\sin^2\vartheta\cos\vartheta & 2\sin^3\vartheta & 2\cos^2\vartheta\sin\vartheta & 2\cos\vartheta\sin^2\vartheta \\ 2\sin^3\vartheta & -2\sin^2\vartheta\cos\vartheta & -2\cos\vartheta\sin^2\vartheta & 2\cos^2\vartheta\sin\vartheta \end{matrix} & \begin{matrix} 2\cos^3\vartheta & 2\cos^2\vartheta\sin\vartheta & 2\sin + \vartheta\cos^2\vartheta & 2\sin^2\vartheta\cos\vartheta \\ 2\cos^2\vartheta\sin\vartheta & 2\cos^3\vartheta & 2\sin^2\vartheta\cos\vartheta & 2\sin\vartheta\cos\vartheta \\ -2\sin\vartheta\cos^2\vartheta & 2\sin^2\vartheta\cos\vartheta & 2\cos^3\vartheta & 2\cos^2\vartheta\sin\vartheta \\ 2\sin^2\vartheta\cos\vartheta & -2\sin\vartheta\cos^2\vartheta & 2\cos^2\vartheta\sin\vartheta & 2\cos^3\vartheta \end{matrix}
\end{pmatrix}_{12\times 12}$$

Proposition (2.12):
If $r\colon \mathrm{Q} \to \mathrm{Q}L(W_5 \otimes ((W_4^* \otimes W_3) \oplus (W_3 \otimes W_2^*))) \otimes W_1^*)$ is
the depiction of Q affects
$(W_5 \otimes ((W_4^* \otimes W_3) \oplus (W_3 \otimes W_2^*))) \otimes W_1^*)$ such that
$r(y) = (r_5(y) \otimes ((r_4^*(y) \otimes r_3(y)) \oplus (r_3(y) \otimes r_2^*(y)) \otimes r_1^*(y)$ for the whole $y \in \mathrm{Q}$, then
$r^*\colon \mathrm{Q} \to (W_5 \otimes ((W_4^* \otimes W_3) \oplus (W_3 \otimes W_2^*))) \otimes W_1^*)$.
Is the depiction of Q affects
$(W_5 \otimes ((W_4^* \otimes W_3) \oplus (W_3 \otimes W_2^*))) \otimes W_1^*)^*$ such that

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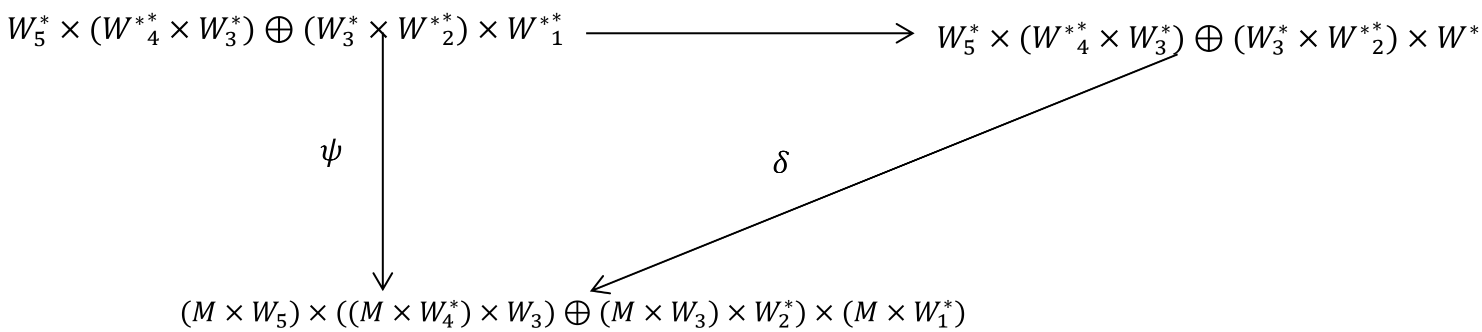
$r^*(y) = (r_5^*(y) \otimes ((r_4(y) \otimes r_3^*(y)) \oplus (r_3^*(y) \otimes r_2(y)) \otimes r_1(y)$
$r^*(y) = \left((r_5(y) \otimes ((r_4^*(y) \otimes r_3(y)) \oplus (r_3(y) \otimes r_2^*(y)) \otimes r_1^*(y)\right)^*$
$r^*(y) = (r_5^*(y) \otimes ((r_4(y) \otimes r_3^*(y)) \oplus (r_3^*(y) \otimes r_2(y)) \otimes r_1(y)$

This being a depiction of Q affects
$(W_5 \otimes ((W_4^* \otimes W_3) \oplus (W_3 \otimes W_2^*))) \otimes W_1^*)^*$



dim $(W_5 \otimes (W_4^* \otimes W_3) \oplus (W_3 \otimes W_2^*) \times W_1^*) = L_i, i = 1,2,3,4,5$
$\Longrightarrow$ dim $(W_5 \otimes (W_4^* \otimes W_3) \oplus (W_3 \otimes W_2^*) \times W_1^*)^*$

where $\dim(W_i) = L_i = 1,2,3,4,5$, and
$r^*(yz) = \left((r_5(yz) \otimes ((r_4^*(yz) \otimes r(yz)) \oplus (r_3(yz) \otimes r_2^*(yz)) \otimes r_1^*(yz)\right)^*$
$= (r_5(yz) \otimes ((r_4^*(yz) \otimes r(yz)) \oplus (r_3(yz) \otimes r_2^*(yz)) \otimes r_1^*(yz)$
$r^*(yz) = \left((r_5(yz) \otimes ((r_4^*(yz) \otimes r_3(yz)) \oplus (r_3(yz) \otimes r_2^*(yz)) \otimes r_1^*(yz)\right)^* \otimes (r_5(z) \otimes ((r_4^*(z) \otimes r_3(z)) \oplus r(z) \otimes r_2^*(z)) \otimes r_1^*(z)) = r^*(y) \otimes r^*(z)$ for all $y, z \in$ Q .

proposition (2.13):
If the (QZA) for Lie group Q on:
Q$L(L_i, M) \cong (W_5 \otimes (W_4^* \otimes W_3) \oplus (W_3 \otimes W_2^*) \times W_1^*)^*$ is:-
$r^*(y) = \left((r_5(y)^{-1} \otimes ((r_4(y))^{tr} \otimes (r_3(y)^{-1}))^{tr} \oplus ((r_3(y))^{tr} \otimes (r_2(y)^{-1})^{tr} \otimes ((r_1(y))^{tr}\right)$ for all $y \in$ Q .
Proof:
$r^*(y) = (r_5(y)^{-1} \otimes (r_4(y)^{-1} \otimes (r_3(y)) \oplus \left((r_3(y) \otimes r_2(y)^{-1})) \otimes (r_1(y)^{-1})\right)^* = (r_5^*(y) \otimes ((r_4^*(y)^{-1} \otimes r_3^*(y)) \oplus (r_3^*(y) \otimes r(y)^{-1}) \otimes r_1^*(y)^{-1} = \left((r_5(y)^{-1})^{tr} \otimes ((r_4(y))^{tr} \otimes (r_3(y)^{-1}))^{tr} \oplus ((r_3(y)^{-1})^{tr} \otimes ((r_1(y))^{tr}\right)$ for all $y \in$ Q . And

$$r^*(yz) = (r(yz)^{-1})^{tr}$$
$$= (r(z^{-1})r(y^{-1}))^{tr}$$
$$= (r(y^{-1})^{tr} r(z^{-1})^{tr})$$
$$= r^*(y) r^*(z), \text{ for all } y, z \in \text{Q} .$$

And on it, r^* is a group homomorphism of Q .

Proposition (2.14):
If $\tilde{r}_i$, $i = 1,2,3,4,$ *and* 5 are depiction of (QZA) for Lie algebra g , and
r_i, 1,2,3,4, *and* 5 are the depiction of (QZA) for Lie groups Q. Such that
$r(e^x) = e^{\tilde{r}(x)}$, for all $x \in$ g and $\tilde{r}(x) = \frac{d}{dt} r(e^{tx})|_{t=0}$, then the relationship between the (QZA)

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for Lie algebras and Lie groups is:

(QZA) for Lie algebra	(QZA) for Lie group
(1) $\tilde{r}_5 \otimes (\tilde{r}_4^* \otimes \tilde{r}_3) \oplus (\tilde{r}_3 \otimes \tilde{r}'^*_2) \otimes \tilde{r}_1^*$	$\frac{d}{dt}(-r_1(e^{tp})\|_{t=0}$ where: $p = \frac{d}{dt}(-r_2(e^{tq})\|_{t=0}$ $q = \frac{d}{dt}(-r_3(e^{tz})\|_{t=0}, z = \frac{d}{dt}(-r_4(e^{ty})\|_{t=0}$ and $y = \frac{d}{dt}(r_5(e^{tx})\|_{t=0}$
(2) $\tilde{r}_5^* \otimes (\tilde{r}_4 \otimes \tilde{r}_3^*) \oplus (\tilde{r}_3^* \otimes \tilde{r}_2) \otimes \tilde{r}_1$	$\frac{d}{dt}(-r_1(e^{tp})\|_{t=0}$ where: $p = \frac{d}{dt}(-r_2(e^{tq})\|_{t=0}$ $q = \frac{d}{dt}(-r_3(e^{tz})\|_{t=0}, z = \frac{d}{dt}(-r_4(e^{ty})\|_{t=0}$ and $y = \frac{d}{dt}(r_5(e^{tx})\|_{t=0}$
(3) $\tilde{r}_5^*(\tilde{r}_4^{**} \otimes \tilde{r}_3^*) \oplus (\tilde{r}_3^* \otimes \tilde{r}_2^{**}) \otimes \tilde{r}_1^{**}$	$\frac{d}{dt}(-r_1(e^{tp})\|_{t=0}$ where: $p = \frac{d}{dt}(-r_2(e^{tq})\|_{t=0}$ $q = \frac{d}{dt}(-r_3(e^{tz})\|_{t=0}, z = \frac{d}{dt}(-r_4(e^{ty})\|_{t=0}$ and $y = \frac{d}{dt}(r_5(e^{tx})\|_{t=0}$

Proposition (2.15):
If $r_i, i = 1,2,3,4, and$ 5 are the depiction of (QZA) for the Lie groups Q ,
and $\tilde{r}_i$, $1,2,3,4, and$ 5 are the depiction of (QZA) for Lie algebras g, therefore the relationship between (QZA) for Lie groups , Lie algebras is:

(QZA) for g	(QZA) for Q
(1)$\tilde{r}(y) = \left(\tilde{r}_5 \otimes (\tilde{r}_4^* \otimes \tilde{r}_3\right) \oplus (\tilde{r}_3 \otimes \tilde{r}_2^*) \otimes \tilde{r}_1^*)(y)$	$\frac{d}{dt}(-r_1(e^{tp})\|_{t=0}$ where: $p = \frac{d}{dt}(-r_2(e^{tq})\|_{t=0}$ $q = \frac{d}{dt}(-r_3(e^{tz})\|_{t=0}, z = \frac{d}{dt}(-r_4(e^{ty})\|_{t=0}$ and $y = \frac{d}{dt}(r_5(e^{tx})\|_{t=0}$
(2)$\tilde{r}(y) = (\tilde{r}_5^* \otimes (\tilde{r}_4 \otimes \tilde{r}_3^*) \oplus (\tilde{r}_3^* \otimes \tilde{r}_2) \otimes \tilde{r}_1)(y)$	$\frac{d}{dt}(-r_1(e^{tp})\|_{t=0}$ where: $p = \frac{d}{dt}(-r_2(e^{tq})\|_{t=0}$ $q = \frac{d}{dt}(-r_3(e^{tz})\|_{t=0}, z = \frac{d}{dt}(-r_4(e^{ty})\|_{t=0}$ and $y = \frac{d}{dt}(r_5(e^{tx})\|_{t=0}$
(3) $\tilde{r}(y) = (\tilde{r}_5^* \otimes (\tilde{r}'^{**}_4 \otimes \tilde{r}_3^*) \oplus (\tilde{r}_3^* \otimes \tilde{r}_2^{**}) \otimes r^{\tilde{\tilde{r}}**}_1)(y)$	$\frac{d}{dt}(-r_1(e^{tp})\|_{t=0}$ where: $p = \frac{d}{dt}(-r_2(e^{tq})\|_{t=0}$ $q = \frac{d}{dt}(-r_3(e^{tz})\|_{t=0}, z = \frac{d}{dt}(-r_4(e^{ty})\|_{t=0}$ and $y = \frac{d}{dt}(r_5(e^{tx})\|_{t=0}$

Proof:
Using $r\ (y) = \frac{d}{dt}(-r_1(e^{yz})|_{t=0}$, for all $y \in$ g thus, $r^*(y) = \frac{d}{dt}(r^*(e^{yz})|_{t=0}$, for all $y \in$ g. Hence $(r_5 \otimes (r_4^* \otimes r_3) \oplus (r_3 \otimes r_2^*) \otimes r_1^*)(y)$

$$= (r_5(e^y) \otimes (r_4^*(e^y) \otimes r(e^y)) \oplus r(e^y) \otimes r_2^*(e^y)) \otimes r_1^*(e^y))$$

$= (\exp\ (\tilde{r}_5(y)) \otimes (\exp\ (\tilde{r}_4^*(y)) \otimes exp\ (\tilde{r}_3^*(y)) \oplus (\exp\ (\tilde{r}_3(y)) \otimes (\exp\ (\tilde{r}_2^{**}(y) \otimes exp\ (\tilde{r}_1^*(y))$.
By the same way, one can prove the parts (2) and (3).

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Conclusion:

n this paper, we presented the fundamental of Lie group , then we showed the properties of representations on Lie group, tensor product and double representations .In our research, we found the relationship between (QZA) of Lie group and tensor product, and by using double representation been studied it on a new structures consisting of five vector spaces , which primarily based on the Schur's lemma.

Finally, the relationship between the (QZA) of Lie group and the Lie algebra has been explained.

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