SOME ADVANCED RESULTS ON CO-HOMOLOGY GROUP OF K(Z,n)

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Abstract:
The purpose of having the serre- spectral sequence is to compute co-homology, in particular when we have a fibration $F \to E \to B$. This study discusses Eilenberg macLane spaces and their co-homology groups. We use the method of spectral sequence to construct the cohomology of K(Z,n), and respectively we can express topological groups of $(n-1)$-connected space X by K(Z,n). The main purpose of this paper is to compute co-homology groups of Eilenberg-MacLane space K(Z,6) and K(Z,7).

Key words: Homology group, Co-homology group, Spectral sequence, Serre spectral sequence.

Introduction:

Algebraic topology is an interesting subject developed from nineteen century, work in complex function theory to do Riemann geometry primarily, and taken up by Henri Poincaŕe. He established a lot of fundamental ideas and further direction for the subject. Algebraic topology roughly is the study of shapes (continuously deformed shapes) and it got a natural connection with differential geometry, algebraic geometry and also with a modern physics. One of the tools that playing fundamental role in algebraic topology is serre spectral sequences, which is apply to fibration and every fibration gives rise to a spectral sequence called the Serre spectral sequence which will describe a very particular relation between the homology groups.

Serre spectral sequences have various applications to algebraic topology theoretically and computationally. (See [6],[7],[8],[9],[10],[11],[12],[13]), in particular a major application is the computation of co-homology group of topological spaces.

Indeed, a main application is the computation of various co-homology groups of topological spaces.

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We know that co-homology groups of K(Z,3) for i < 14 computed, (see [2]). Furthermore, the co-homology group of K(Z,4) and K(Z,5) computed (see [5] and [1]). In this paper first we present the construction of spectral sequence, we will also present how to compute the co-homology groups of K(Z,6) and K(Z,7) with coefficient group Z. The main references for any definitions or theorems are Allen Hatcher's Algebraic topology (see [1]). The more topological pieces of this exposition follow McCleary, a user's guide to spectral sequences (see [4]). The rest of the paper is arranged as follows: The second section, is an illustration of the basic notions and definitions from algebraic topology and some theorems which will be used later in the particular case of other sections. All of these preliminaries are based on the references [1], [2], [4] and [5]. Section three presents a brief discussion about spectral sequences and a short introduction to the spectral sequences for co-homology for a fibration F → X → B. The Section four (main section) of this paper is devoted to Co-homology Groups of K(Z,6) Via Fibrations. Moreover, our explanation of this section includes some lemma and then prove them by using spectral sequence, to compute the co-homology groups of K(Z,6). The penultimate and another main section deals with calculating K(Z,7) with some interesting lemma. At the end of this paper, we have mentioned strong application of this space and K(Z,n) .

Theorems and Definitions from Algebraic Topology

The standard theorem and definitions from algebraic topology that is crucial in this article have been stated. The followings are the key definitions of this study.

Definition 2.1. A map which has a homotopy lifting property with respect to all path connected spaces, and defined as $P : E \to B$ is called fibration (see [3])

Definition 2.2. An Eilenberg- Maclane space $K(G,n)$, is a space X which has just one non-trivial homotopy group. (See [1])

Definition 2.3. A topological space is called simply- connected if it is path- connected and has trivial fundamental group. (See [2])

Definition 2.4. A space with basepoint x0 is said to be n- connected if $\pi i(X, x0) = 0 \ \forall\, i < n$. (see[4])

Theorem 2.1. (Hurewicz) If space X is (n − 1)- connected for $n \geq 2$, then $Hi(X) = 0 \ for\ i < n$ and $\pi n(X) \simeq Hn(X)$. If a pair (X, A) is (n−1)- connected, $n \geq 2$, with A simply- connected and nonempty, then $Hi(X, A) \simeq \pi i(X, A) = 0$ for $i < n$ and $\pi n(X, A) \simeq Hn(X, A)$. (see [1])

Theorem 2.2. (Universal Coefficients for Co-homology) If a chian complex C of free abelian groups has homology groups $Hi(C)$,Then the cohomology groups $H^{i}(C, G)$ of cochain complex $Hom(Ci, G)$ are determined by split exact sequences

$0 \rightarrow Ext(Hn - 1(X), G) \rightarrow H^n(C, G) \rightarrow Hom(Hn(C), G) \rightarrow$
0. …………………..(2.1)

In practice, the Ext term either vanishes or is computable.(see [4], [5], and [14])

Theorem 2.3. **(Homology with Coefficient)**

If the homology groups Hn and H_{n-1} of a chain complex C of free abelian groups are finitely generated, with torsion Subgroups $Tn \subset Hn$ and $T_{n-1} \subset H_{n-1}$,then

$H^n(C, Z) \simeq (\frac{Hn}{Tn}) \oplus T_{n-1}$.

In practice, the Tor term either vanishes or is easily computable. (see [1])

Theorem 2.4. If C is a chain complex of free abelian groups, then there are natural short exact sequences

$0 \rightarrow Hn(C) \otimes G \rightarrow Hn(C, G) \rightarrow Tor(H_{n-1}(C), G) \rightarrow 0$ (see [2]). ………………(2.2)

Spectral Sequences

The tool to compute the co-homology group of a chain complex called Spectral sequence. It results from a filtration of the dual chain complex and it gives an alternative method to determine the co-homology of the dual chain complex (see[4]). A spectral sequence made up of pages, which is a sequence of intermediate dual chain complexes pages are denotes as $E_0, E_1, E_2, E_3 \ldots$, and differentials denoted by $d_0, d_1, d_2, d_3, \ldots$, where E_{n+1} is the co-homology of Er.

This limit process may converge, in which case the limit page is denoted by E_∞. Even if there is convergence to E_∞, reconstruction is still needed to obtain H from E_∞ Although the differentials dr's cannot always be all computed, the existence of the spectral sequence often reveals deep facts about the dual chain complex. The spectral sequence and its internal mechanisms can still lead to very useful and deep applications (see[4], [14])

Theorem 3.1. (Spectral Sequences for Cohomology)

For a fibration $F \rightarrow X \rightarrow B$ with B path-connected and $\pi_1(B)$ acting trivially

on $H^*(F, G)$, there is a spectral sequences $\{E_r^{p,q}, d_r\}$ with:

1. $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ and $E_{r+1}^{p,q} = \frac{ker\ d_r}{Img\ d_r}$ at $E_r^{p,q}$.
2. Stable terms $E_\infty^{p,n-p}$ is isomorphic to the successive quotients $\frac{F_p^n}{F_{p+1}^n}$ in a filtration $0 \subset F_n^n \subset F_{n-1}^n \subset \ldots \subset F_0^n = H^n(X,G)$ of $H^n(X,G)$.
3. $E_2^{p,q} \approx H^p(B, H^q(F,G))$ $(see[3])$.

Co-homology Groups of K(Z,6) Via Fibrations

For any path connected space $(X, x0)$ in algebraic topology, there is a path fibration

$$\Omega X \to P X \to X,$$

where X is the base space, PX is the total contractible space and ΩX is the fiber space over the base space $(X, x0)$, which called loop space.

Now Suppose that space $X = K(Z,6)$, is the base space, then there is a path space fibration

$$K(Z,5) = \Omega K(Z,6) \to PK(Z,6) \to K(Z,6).$$

The topological space $K(Z,6)$ is (5 -connected), by defintion n- connected space and Hurewicz theorem, we have

$$H_1(K(Z,6),Z) \simeq H_2(K(Z,6),Z) \simeq H_3(K(Z,6),Z) \simeq H_4(K(Z,6),Z) \simeq H_5(K(Z,6),Z) \simeq 0 \quad \ldots\ldots\ldots(4.1)$$

Now with the using of theorem (2.2), it is easy to show that

$$H^1(K(Z,6),Z) \simeq H^2(K(Z,6),Z) \simeq H^3(K(Z,6),Z) \simeq H^4(K(Z,6),Z) \simeq H^5(K(Z,6),Z) \simeq 0 \quad \ldots\ldots..(4.2)$$

On the other hand by Hurewicz theorem we see that $\pi_6(K(Z,6)) \simeq H_6(K(Z,6),Z) \simeq Z.$

Then by using universal coefficient theorem we have:

$$H^6(K(Z,6),Z) \simeq Ext(H_5(K(Z,6),Z)) \oplus Hom(H_6(K(Z,6),Z))\, Z. \quad \ldots\ldots\ldots\ldots\ldots.....(4.3)$$

Now by setting $E_2^{p,q} := \mathrm{H^p(K(Z,6), H^q(K(Z,5))}$

And utilizing theorems (2.1), (2.2) and co-homology groups of topological space $K\ (Z, 5)$ (see[6]), There are following results:

$$E_2^{0,0} = H^0(K(Z,6), H^0(K(Z,5),Z)) = H^0(K(Z,6),Z) \simeq Z$$

$$E_2^{0,1} = H^0(K(Z,6), H^1(K(Z,5),Z)) = H^0(K(Z,6),0) \simeq 0$$

$$E_2^{0,2} = H^0(K(Z,6), H^2(K(Z,5),Z)) = H^0(K(Z,6),0) \simeq 0$$

$$E_2^{0,3} = H^0(K(Z,6), H^3(K(Z,5),Z)) = H^0(K(Z,6),0) \simeq 0$$

$$E_2^{0,4} = H^0(K(Z,6), H^4(K(Z,5),Z)) = H^0(K(Z,6),0) \simeq 0$$

$$E_2^{0,5} = H^0(K(Z,6), H^5(K(Z,5),Z)) = H^0(K(Z,6),Z) \simeq Z$$

$$E_2^{0,6} = H^0(K(Z,6), H^6(K(Z,5),Z)) = H^0(K(Z,6),0) \simeq 0$$

$$E_2^{0,7} = H^0(K(Z,6), H^7(K(Z,5),Z)) = H^0(K(Z,6),0) \simeq 0$$

$$E_2^{0,8} = H^0\big(K(Z,6), H^8(K(Z,5),Z)\big) = H^0(K(Z,6),0) \simeq 0$$

$$E_2^{0,9} = H^0\big(K(Z,6), H^9(K(Z,5),Z)\big) = H^0(K(Z,6),0) \simeq 0$$

$$E_2^{0,10} = H^0(K(Z,6), H^{10}(K(Z,5),Z)) = H^0(K(Z,6),Z_6) \simeq Z_6 .$$

$$E_2^{1,0} = H^1(K(Z, 6), H^0(K(Z, 5), Z)) = H^1(K(Z, 6), Z) \simeq 0$$

$$E_2^{1,2} = H^1(K(Z, 6), H^2(K(Z, 5), Z)) = H^1(K(Z, 6), 0) \simeq 0$$

$$E_2^{1,3} = H^1(K(Z, 6), H^3(K(Z, 5), Z)) = H^1(K(Z, 6), 0) \simeq 0$$

$$E_2^{1,4} = H^1(K(Z, 6), H^4(K(Z, 5), Z)) = H^1(K(Z, 6), 0) \simeq 0$$

$$E_2^{1,5} = H^1(K(Z, 6), H^5(K(Z, 5), Z)) = H^1(K(Z, 6), Z) \simeq 0$$

$$E_2^{1,6} = H^1(K(Z, 6), H^6(K(Z, 5), Z)) = H^1(K(Z, 6), 0) \simeq 0$$

$$E_2^{1,7} = H^1(K(Z, 6), H^7(K(Z, 5), Z)) = H^1(K(Z, 6), 0) \simeq 0$$

$$E_2^{1,8} = H^1(K(Z, 6), H^8(K(Z, 5), Z)) = H^1(K(Z, 6), 0) \simeq 0$$

$E_2^{1,9} = H^1(K(Z,6), H^9(K(Z,5),Z)) = H^1(K(Z,6),0) \simeq 0$

$E_2^{2,0} = H^2(K(Z,6), H^0(K(Z,5),Z)) = H^2(K(Z,6),Z) \simeq 0$

$E_2^{2,1} = H^2(K(Z,6), H^1(K(Z,5),Z)) = H^2(K(Z,6),0) \simeq 0$

$E_2^{2,2} = H^2(K(Z,6), H^2(K(Z,5),Z)) = H^2(K(Z,6),0) \simeq 0$

$E_2^{2,3} = H^2(K(Z,6), H^3(K(Z,5),Z)) = H^2(K(Z,6),Z) \simeq 0$

$E_2^{2,4} = H^2(K(Z,6), H^4(K(Z,5),Z)) = H^2(K(Z,6),0) \simeq 0$

$E_2^{2,5} = H^2(K(Z,6), H^5(K(Z,5),Z)) = H^2(K(Z,6),Z) \simeq 0$

$E_2^{2,6} = H^2(K(Z,6), H^6(K(Z,5),Z)) = H^2(K(Z,6),0) \simeq 0$

$E_2^{2,7} = H^2(K(Z,6), H^7(K(Z,5),Z)) = H^2(K(Z,6),0) \simeq 0$

$E_2^{2,8} = H^2(K(Z,6), H^8(K(Z,5),Z)) = H^2(K(Z,6),0) \simeq 0$

$E_2^{2,9} = H^2(K(Z,6), H^9(K(Z,5),Z)) = H^2(K(Z,6),0) \simeq 0$

$E_2^{3,0} = H^3(K(Z,6), H^0(K(Z,5),Z)) = H^3(K(Z,6),Z) \simeq 0$

$E_2^{3,1} = H^3(K(Z,6), H^1(K(Z,5),Z)) = H^3(K(Z,6),0) \simeq 0$

$E_2^{3,2} = H^3(K(Z,6), H^2(K(Z,5),Z)) = H^3(K(Z,6),0) \simeq 0$

$E_2^{3,3} = H^3(K(Z,6), H^3(K(Z,5),Z)) = H^3(K(Z,6),Z) \simeq 0$

$E_2^{3,4} = H^3(K(Z,6), H^4(K(Z,5),Z)) = H^3(K(Z,6),0) \simeq 0$

$E_2^{3,5} = H^3(K(Z,6), H^5(K(Z,5),Z)) = H^3(K(Z,6),Z) \simeq 0$

$E_2^{3,6} = H^3(K(Z,6), H^6(K(Z,5),Z)) = H^3(K(Z,6),0) \simeq 0$

$E_2^{3,7} = H^3(K(Z,6), H^7(K(Z,5),Z)) = H^3(K(Z,6),0) \simeq 0$

$E_2^{3,8} = H^3(K(Z,6), H^8(K(Z,5),Z)) = H^3(K(Z,6),0) \simeq 0$

$E_2^{3,9} = H^3(K(Z,6), H^9(K(Z,5),Z)) = H^3(K(Z,6),0) \simeq 0$

$$E_2^{4,0} = H^4(K(Z,6), H^0(K(Z,5),Z)) = H^4(K(Z,6),Z) \simeq 0$$

$$E_2^{4,1} = H^4(K(Z,6), H^1(K(Z,5),Z)) = H^1(K(Z,6),0) \simeq 0$$

$$E_2^{4,2} = H^4(K(Z,6), H^2(K(Z,5),Z)) = H^4(K(Z,6),0) \simeq 0$$

$$E_2^{4,3} = H^4(K(Z,6), H^3(K(Z,5),Z)) = H^4(K(Z,6),0) \simeq 0$$

$$E_2^{4,4} = H^4(K(Z,6), H^4(K(Z,5),Z)) = H^4(K(Z,6),0) \simeq 0$$

$$E_2^{4,5} = H^4(K(Z,6), H^5(K(Z,5),Z)) = H^4(K(Z,6),Z) \simeq 0$$

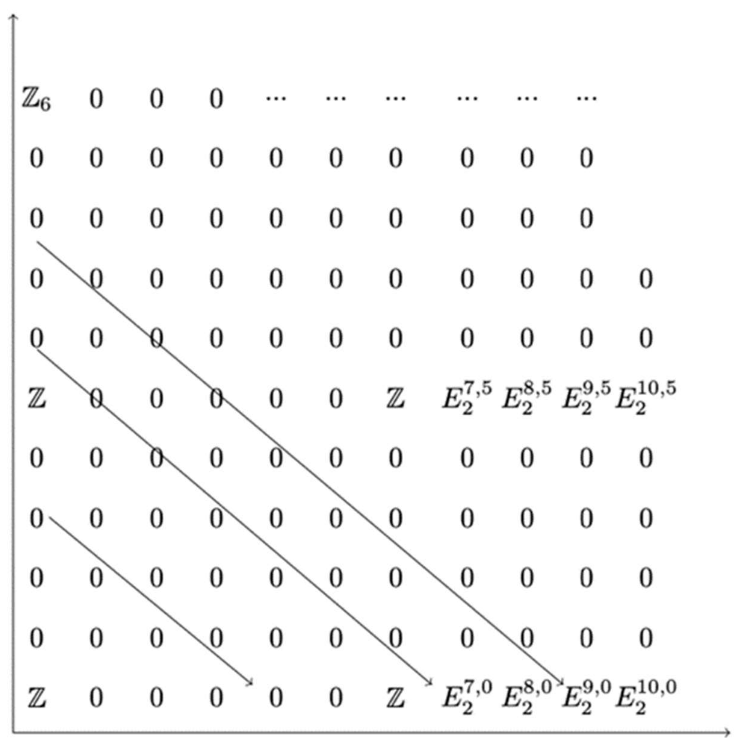
$$E_2^{4,6} = H^4(K(Z,6), H^6(K(Z,5),Z)) = H^4(K(Z,6),0) \simeq 0$$

$$E_2^{4,7} = H^4(K(Z,6), H^7(K(Z,5),Z)) = H^4(K(Z,6),0) \simeq 0$$

$$E_2^{4,8} = H^4(K(Z,6), H^8(K(Z,5),Z)) = H^4(K(Z,6),0) \simeq 0$$

$$E_2^{4,9} = H^4(K(Z,6), H^9(K(Z,5),Z)) = H^4(K(Z,6),0) \simeq 0$$

Addition to the above result, One can compute page 2 as follows:



Since of topological space $PK(Z,6)$ is contractible, then for any p and q, $E_{\infty}^{p,q}$

converges to $H^{p+q}(PK(Z,6),Z)$, In the other words, we have $E_{\infty}^{p,q} \Rightarrow H^{p+q}(PK(Z,6),Z) = 0$.

Lemma 4.1. $E_2^{5,0} \Rightarrow H^5(K(Z,6),Z) = 0$.

Proof. Since every differential map in page 2 and other pages, comes in and out $E_2^{5,0}$ is zero, so $E_2^{5,0}$will be remained to page infinity and converges to then $E_{\infty}^{5,0} = 0,$ then

$$E_2^{5,0} = H^5(K(Z,6),Z) = 0.$$

Lemma 4.2. $E_2^{7,0} \Rightarrow H^6(K(Z,6),Z) = 0$.

Proof. Similarly we use the same method, we have$E_2^{6,0} \Rightarrow H^6(K(Z,6),Z) = 0$

Lemma 4.3. $E_2^{12,0} = H^{12}(K(Z,6),Z) = Z_6$.

Proof. A spectral sequence $\{E_r^{p,q}, d_r\}$ is a collection of vector space or groups

$E_r^{p,q}$ equipped with differential map d_r (i.e. $d_r \ o \ d_r = d_r^2 = 0$)

$$E_{r+1}^{p,q} = H(E_r^{p,q}, d_r)$$

Now consider following sequence, such that $d_2^2 = 0$

$0 \to Z_6 \simeq E_2^{0,10} \to E_2^{6,5} \simeq Z \to E_2^{12,0} \to 0$. …………..(4.4)

The generator of $Z6$ has finite order but there is no finite order element in Z then, from Hom algebraic properties, there is no linear map from $Z6$ to Z. So the left differential map in above exact sequence should be zero, but because of exactness and converging properties, the right map should be surjective.

Consider that right differential map not be an isomorphism, so we will have non

trivial subgroup of $E_2^{12,0}$, such that converges to $E_{\infty}^{12,0}$= 0, which will not be true.

So we have

$E_2^{12,0} = H^{12}(K(Z,6),Z) \simeq Z_6$.

The following is the last result about this section:

Corollary 4.1.

$$E_2^{1,0} = H^1(K(Z,6), H^0(K(Z,5),Z)) = H^1(K(Z,6),Z) \simeq 0$$

$$E_2^{2,0} = H^2(K(Z,6), H^0(K(Z,5),Z)) = H^2(K(Z,6),Z) \simeq 0$$

$$E_2^{3,0} = H^3(K(Z,6), H^0(K(Z,5),Z)) = H^3(K(Z,6),Z) \simeq 0$$

$$E_2^{4,0} = H^4(K(Z,6), H^0(K(Z,5),Z)) = H^4(K(Z,6),Z) \simeq 0$$

$$E_2^{5,0} = H^5(K(Z,6), H^0(K(Z,5),Z)) = H^5(K(Z,6),Z) \simeq 0$$

$$E_2^{6,0} = H^6(K(Z,6), H^0(K(Z,5),Z)) = H^6(K(Z,6),Z) \simeq Z$$

$$E_2^{7,0} = H^7(K(Z,6), H^0(K(Z,5),Z)) = H^7(K(Z,6),Z) \simeq 0$$

$$E_2^{8,0} = H^8(K(Z,6), H^0(K(Z,5),Z)) = H^8(K(Z,6),Z) \simeq 0$$

$$E_2^{9,0} = H^9(K(Z,6), H^0(K(Z,5),Z)) = H^9(K(Z,6),Z) \simeq 0$$

$$E_2^{10,0} = H^{10}(K(Z,6), H^0(K(Z,5),Z)) = H^{10}(K(Z,6),Z) \simeq 0$$

$$E_2^{11,0} = H^{11}(K(Z,6), H^0(K(Z,5),Z)) = H^{11}(K(Z,6),Z) \simeq 0$$

$$E_2^{12,0} = H^{12}\big(K(Z,6), H^0(K(Z,5),Z)\big) = H^{12}(K(Z,6),Z) \simeq Z_6$$

Cohomology Groups of K(Z,7) Via Fibration

Now consider base space X = K (Z, 7), then

$$K(Z,6) = \Omega K(Z,7) \to PK(Z,7) \to K(Z,7).$$

Here $K(Z,7)$ is (6 - connected), and as we showed above, we obtain the following results

$$H_1(K(Z,7),Z) \simeq H_2(K(Z,7),Z) \simeq H_3(K(Z,7),Z) \simeq H_4(K(Z,7),Z) \simeq H_5(K(Z,7),Z) \simeq H_6(K(Z,7),Z) \simeq 0 \qquad \text{……….(5.1)}$$

$$H^1(K(Z,7),Z) \simeq H^2(K(Z,7),Z) \simeq H^3(K(Z,7),Z) \simeq H^4(K(Z,7),Z) \simeq H^5(K(Z,7),Z) \simeq H^6(K(Z,7),Z) \simeq 0 \qquad \text{………..(5.2)}$$

And $H^7(K(Z,7),Z) \simeq Ext(H_6(K(Z,7),Z)) \oplus Hom(H_7(K(Z,7),Z)) \simeq Z.$

Now it is time to calculate higher co-homology groups and respectively homology

Groups of $K(Z,7)$. By setting

$$E_2^{p,q} := H^p(K(Z,7); H^q(K(Z,6))$$

there are following results which are proved.

$$E_2^{0,0} = H^0(K(Z,7), H^0(K(Z,6),Z)) = H^0(K(Z,7),Z) \simeq Z$$

$$E_2^{0,1} = H^0(K(Z,7), H^1(K(Z,6),Z)) = H^0(K(Z,7),0) \simeq 0$$

$$E_2^{0,2} = H^0(K(Z,7), H^2(K,Z,6),Z)) = H^0(K(Z,7),0) \simeq 0$$

$$E_2^{0,3} = H^0(K(Z,7), H^3(K(Z,6),Z)) = H^0(K(Z,7),0) \simeq 0$$

$$E_2^{0,4} = H^0(K(Z,7), H^4(K(Z,6),Z)) = H^0(K(Z,7),0) \simeq 0$$

$$E_2^{0,5} = H^0(K(Z,7), H^5(K(Z,6),Z)) = H^0(K(Z,7),0) \simeq 0$$

$$E_2^{0,6} = H^0(K(Z,7), H^6(K(Z,6),Z)) = H^0(K(Z,7),Z) \simeq Z$$

$$E_2^{0,7} = H^0(K(Z,7), H^7(K(Z,6),Z)) = H^0(K(Z,7),0) \simeq 0$$

$$E_2^{0,8} = H^0(K(Z,7), H^8(K(Z,6),Z)) = H^0(K(Z,7),0) \simeq 0$$

$$E_2^{0,9} = H^0(K(Z,7), H^9(K(Z,6),Z)) = H^0(K(Z,7),0) \simeq 0$$

$$E_2^{0,10} = H^0(K(Z,7), H^{10}(K(Z,6),Z)) = H^0(K(Z,7),0) \simeq 0$$

$$E_2^{0,11} = H^0(K(Z,7), H^{11}(K(Z,6),Z)) = H^0(K(Z,7),0) \simeq 0$$

$$E_2^{0,12} = H^0(K(Z,7), H^{12}(K(Z,6),Z)) = H^0(K(Z,7),Z_6) \simeq Z_6$$

Then for all $i \in [1,2,3,4,5,6]$, we have

$$E_2^{i,0} = H^2(K(Z,7), H^0(K(Z,6),Z)) = H^2(K(Z,7),Z) \simeq 0$$

$$E_2^{i,1} = H^2(K(Z,7), H^1(K(Z,6),Z)) = H^2(K(Z,7),0) \simeq 0$$

$$E_2^{i,2} = H^2(K(Z,7), H^2(K(Z,6),Z)) = H^2(K(Z,7),0) \simeq 0$$

$$E_2^{i,3} = H^2(K(Z,7), H^3(K(Z,6),Z)) = H^2(K(Z,7),0) \simeq 0$$

$$E_2^{i,4} = H^2(K(Z,7), H^4(K(Z,6),Z)) = H^2(K(Z,7),0) \simeq 0$$

$$E_2^{i,5} = H^2(K(Z,7), H^5(K(Z,6),Z)) = H^2(K(Z,7),0) \simeq 0$$

$$E_2^{i,6} = H^2(K(Z,7), H^6(K(Z,6),Z)) = H^2(K(Z,7),0) \simeq 0$$

$$E_2^{i,7} = H^2(K(Z,7), H^7(K(Z,6),Z)) = H^2(K(Z,7),0) \simeq 0$$

$$E_2^{i,8} = H^2(K(Z,7), H^8(K(Z,6),Z)) = H^2(K(Z,7),0) \simeq 0$$

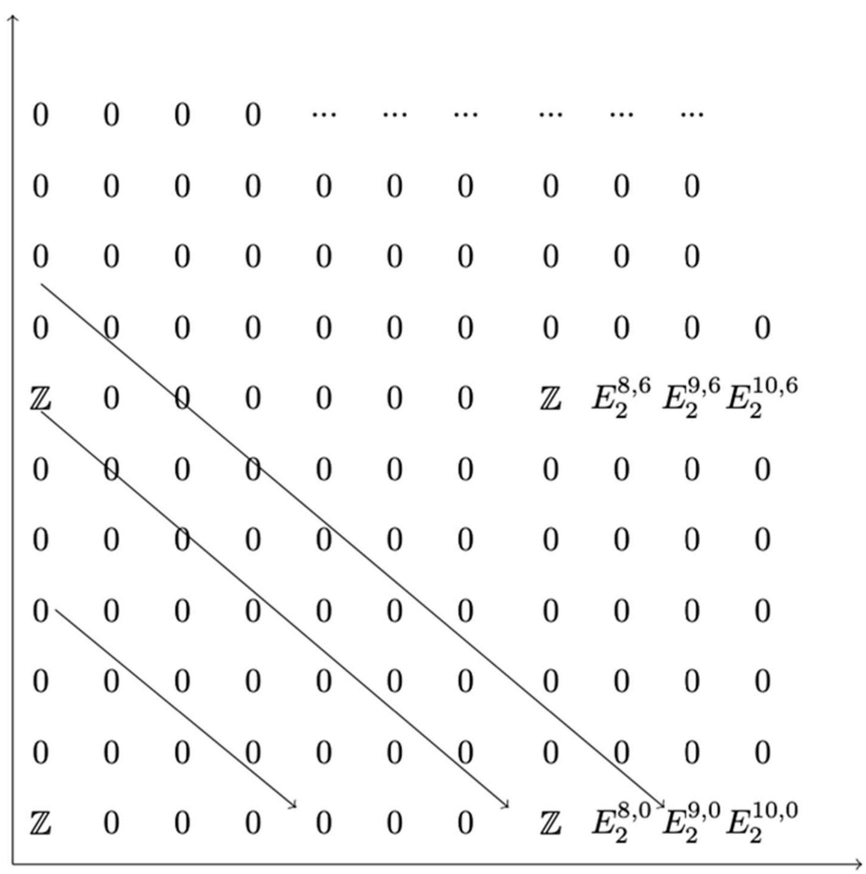
$$E_2^{i,9} = H^2(K(Z,7), H^9(K(Z,6),Z)) = H^2(K(Z,7),0) \simeq 0$$

Lemma 5.1.

$$E_2^{1,9} = E_2^{2,9} = E_2^{3,9} = E_2^{4,9} = E_2^{5,9} = E_2^{6,9} = 0$$

Proof. The proof of this lemma is obvious by straight calculation.

By considering page 2 as follow:



Lemma 5.2. $E_2^{6,0} = H6(K(Z,7),Z) = 0$.
Proof. Each differential map which come and go out $E_2^{6,0}$ are zero. Then $E_2^{6,0}$ will remain to page infinity and converge to $E_\infty^{6,0} = 0$, therefore $E_2^{6,0} = H6(K(Z,7),Z) = 0$

Lemma 5.3. $E_2^{7,0} = H7(K(Z, 7), Z) = Z$
Proof. proof by Hurewicz theorem is clear.
Lemma 5.4. $E_2^{8,0} = H^8(K(Z,7),Z) \simeq 0$
Proof. As we know, $E_2^{(8,)0}$ should be eliminated by the differential map, and thus $E_2^{8,0} \simeq 0$.
Lemma 5.5. $E_2^{9,0} = E_2^{10,0} = E_2^{11,0} = E_2^{12,0} = E_2^{13,0} \simeq 0$.
Proof. For each $i \in [9,10,11,12,13]$ and $E_2^{(i,0)}$, it is easy to see that $E_2^{(i,0)} \simeq 0$.
Lemma 5.6. $E_2^{14,0} \simeq Z_6$.
Proof. From the page 2, and following a short exact sequence
$0 \to Z_6 \xrightarrow{g} Z \xrightarrow{f} E_2^{14,0} \to 0$
(Here($f \circ g=0$), and g is a map which multipliable by 6), thus $E_2^{14,0} \simeq Z_6$, so $E_2^{10,0} \simeq Z_6$.

Application

Most mathematicians were interested to find the co-homology and homology groups of arbitrary (n − 1)- connected topological space X . One of the significant way to look deeply at these groups are using co-homology and homology groups of K(Z,n). In other words, co-homology and homology groups of X can be expressed by co-homology groups of K(Z,n) (see [1], [3]). Here we have mentioned relation between topological space X and K(Z,n).
If X be an arbitrary (n − 1)- connected topological space, then by Hurewiz map $h: \pi n(X) \to Hn(X)$ is an isomorphism. Also by using universal coefficient theorem,
$Hn(X) \simeq Hom(Hn(X),Z) \oplus Ext(H_{(n-1)}(X),Z) = Hom(Hn(X),Z)$
Theorem 6.1. For any arbitrary (n − 1)- connected topological space X, there is an natural isomorphism,
$H^n(X) \cong [X,K(Z,n)]$
Lets consider following serre fibration for topological space K(Z,n),
$K(Z,n-1) \to E = P K(Z,n) \to K(Z,n)$
If $f: X \to K(Z,n)$ and $p: E \to K(Z,n)$, then by using pullback property,
there is exist a following fibration for X,
$K(Z,n-1) \to E^{**} \to X$.
Here $E^{**} = \{(x,e) | f(x) = p(e), x \in X, e \in E\}$. Therefore, homology and co-homology of X can be expressed by homology and cohomology of K(Z,n - 1) and respectively.

References:
Allen H. Algebraic Topology, Cambridge University Press, 2001.
Allen H. Spectral sequences in Algebraic Topology (unfinished, online)

John M. A user’s guide to spectral sequences, Second edition. Cambridge Studies in Advanced Mathematics, 58. Cambridge University Press, Cambridge, 2001.

Peter. M. A concise course in algebraic topology. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999.

Keyvan S. and Hero S. Cohomology Group of K(Z,4). Diyala Journal for pure Science, January 2020.

Keyvan S. Cohomology Group of K(Z, 4) and K(Z, 5). Arxiv:2106.05448v1[math.AT] 10 June 2021.

Robert M. and Martin T., Cohomology Operations and Applications in Homotopy Theory, Harper & Row, Publishers, 1968

Eldon D. and Vasquez A. T. Some small aspherical spaces, J. Austral. Math. Soc. 16 (1973),

William D. and C. W. Wilkerson, A new finite loop space at the prime two, Journal A.M.S. 6 (1993), 37–63.

William D. and Haynes M. and Clarence W. Homotopical uniqueness of classifying spaces, Topology 31 (1992), 29–45.

Clliford H. D. Topology of metric complexes, Am. J. Math. 74 (1952), 555–577.

Alberecht D. and Rene T. Quasifaserungen und unendliche symmetrische Produkte, Ann. Of Math. 67 (1958), 239–281.

Richard M. A survey of the spherical space form problem, Math. Reports 2 (1985), 223–283, Harwood Acad. Pub.

Armand B., Sur la cohomologie des espaces fibr es principaux, Ann. of Math. 57 (1953), 115–207.

Raoul B. and Hans S. On the Pontryagin product in spaces of paths, Comment. Math. Helv. 27 (1953), 320–337.