Volume 26 Issue 02, 2023 ISSN: 1005-3026 https://dbdxxb.cn/ Original Research Paper

COUPLED FIXED POINT THEOREMS FOR COMPATIBLE MAPPINGS IN PARTIALLY ORDERED G-METRIC SPACES

Uma. M
Government first grade college, Kolar -563101,Karnataka

M.Vijaya Kumar
Bhopal

Abstract
In this present paper, we prove G- metric space for coupled coincidence and coupled common fixed point theorems for compatible mappings in partially ordered. The results on fixed point theorems are generalizations of some existing results. We also give an example to support our results.

Keywords: partially ordered set, couple coincidence point, coupled fixed point, compatible mappings, G-metric space.
MSC: 47H10, 54H25.

1. Introduction :

It is well known that fixed point theory is one of the most powerful and fruitful tools in nonlinear analysis, differential equation, and economic theory and has been studied in many various metric spaces. Especially, in 2006, Mustafa and Sims [15] introduced a generalized metric spaces which are called G-metric space. Follow Mustafa and Sims' work, many authors developed and introduced various fixed point theorems in G-metric spaces [7,]. Simultaneously, fixed point theory has developed rapidly in partially ordered metric spaces.Fixed point theorems have also been considered in partially ordered probabilistic metric spaces [9], in partially ordered cone metric spaces [1], and in partially ordered G-metric spaces [2]. In particular, in [4], Bhaskar and Lakshmikantham introduced notions of a mixed monotone mapping and a coupled fixed point, proved some coupled fixed point theorems for mixed monotone mappings, and discussed the existence and unique of solutions for periodic boundary value problems. Afterwards, some coupled fixed point and coupled coincidence point results and their applications have been established.
In this paper, we prove coupled coincidence and coupled common fixed point theorems for compatible mappings in partially ordered G-metric spaces. The results on fixed point theorems are generalizations of some existing results. We give an example to illustrate that our result is better than the results of Aydi at al. [3].
Note: let N denote the set of nonnegative integers, and R^+ be the set of positive real numbers.

2. PRELIMINARIES

Before giving our main results, we recall some basic concepts and results in G-metric spaces.

Submitted: 28/07/2023 **Accepted**: 25/08/2023

Definition 2.1:

Let X be a non-empty set, G : X × X × X → R^+ be a function satisfying the following properties:

(G1) G(x, y, z) = 0 if x = y = z.

(G2) 0 < G(x, x, y) for all x, y ∈ X with x ≠y.

(G3) G(x, x, y) ≤ G(x, y, z) for all x, y, z ∈ X with y ≠z.

(G4) G(x, y, z) = G(x, z, y) = G(y, z, x) = . . .(symmetry in all three variables).

(G5) G(x, y, z) ≤ G(x, a, a) + G(a, y, z) for all x, y, z, a ∈ X (rectangle inequality).

Then the function G is called a generalized metric and the pair (X, G) is called a G-metric space.

Example 2.2 :

Let X be nonempty set and E the nonempty set of parameters . We G: XxX xX → R by

$$G(x,y,z) = \begin{cases} 0 \\ 1 \\ 2 \end{cases}$$

0 if all of the variables are equal, 1 = if two of the variable are equal , and the remaining one is distinct:2 = if all of the variables are distinct For all x,y,z, belongs to X . Then G sastisfies all G- metric space

Definition 2.3.

Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence of points of X.

I. A point x ∈ X is said to be the limit of the sequence $\{x_n\}$ if $\lim_{n,m\to\infty} G(x_n, x_n, x_m) = 0$, and one says the sequence $\{x_n\}$ is G-convergent to x.

II. if $x_n \to x$ in G-metric space (X, G) then, for any > 0, there exists a positive integer N such that $G(x_n, x_n, x_m) < \in$ for all n, m > N.

III. A sequence $\{x_n\}$ is called G-Cauchy if every ∈ > 0, there exists a positive N such that $G(x_n, x_n, x_m) < \in$ for all n, m, l > N, that is, if $G(x_n, x_m, x_l) < \in \to 0$, as n, m, l → ∞.

In [1], the authors have shown that the G-metric induces a Hausdorff topology, and the convergence described in the above definition is relative to this topology. The topology being Hausdorff, a sequence can converge at most to a point.

Lemma 2.4.

If (X, G) is a G-metric space, then the following are equivalent.

(1) $\{x_n\}$ is G-convergent to x.

(2) $G(x_n, x_n, x_m) < \in \to 0$ as n → ∞.

(3) $G(x_n, x, x) \to 0$ as n → ∞.

(4) $G(x_m, x_n, x) \to 0$ as m, n → ∞.

Lemma 2.5.

If (X, G) is a G-metric space, then the following are equivalent.

(1) The sequence $\{x_n\}$ is G-Cauchy.

(2) For every $\in > 0$, there exists a positive integer N such that $G(x_n, x_n, x_m) < \in$ for all n, m > N.

Lemma 2.6 :

If (X, G) is a G-metric space,

I. then $G(x, y, y) \leq 2G(y, x, x)$ for all $x, y \in X$.
II. then $G(x, x, y) \leq G(x, x, z)+G(z, z, y)$ for all $x, y, z \in X$.
III. Then the function G(x, y, z) is jointly continuous in all three of its variables

Definition 2.7:

Let (X, G), (X', G') be two G-metric spaces. Then a function $f : X \to X_0$ is Gcontinuous at a point $x \in X$ if and only if it is G-sequentially continuous at x; that is, whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is G' -convergent to f(x).

Definition 2.8:

A G-metric space (X, G) is said to be G-complete (or a complete G-metric space) if every G-Cauchy sequence in (X, G) is convergent in X.

Next, we need some notions about partially ordered set.

Definition 2.9:

Let $(X, \preccurlyeq)$ be a partially ordered set and let $F : X \times X \to X$. The mapping F is said to have the mixed monotone property if F(x, y) is monotone non-decreasing in x and is monotone non-increasing in y; that is, for any $x, y \in X$,

$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$ and

$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2)$.

Definition 2.10:

An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if

$$x = F(x, y) \text{ and } y = F(y, x).$$

Definition 2.11:

Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$ be two mappings. We say that F has the mixed-g-monotone property if F(x, y) is g-monotone nondecreasing in x and it is g-monotone nonincreasing in y, that is, for any $x, y \in X$, we have:

$x_1, x_2 \in X, g(x_1) \leq g(x_2) \Rightarrow F(x_1, y) \leq F(x_2, y)$ and, respectively,

$y_1, y_2 \in X, g(y_1) \leq g(y_2) \Rightarrow F(x, y_1) \geq F(x, y_2)$.

Definition 2.12:

An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mapping $F : X \times X \to X$ and $g : X \to X$ if

$gx = F(x, y)$ and $gy = F(y, x)$. for all $x, y \in X$

Definition 2.13:

We say that the mapping $F : X \times X \to X$ and $g : X \to X$ are commutative if $g(F(x, y)) = F(gx, gy)$ for all $x, y \in X$. in[12] Lakshmikantham and Ciri´c considered the following class of functions. We denote by Φ the set ´ of functions $\phi : [0, +\infty) \to [0, +\infty)$ satisfying

(a) $\varphi^{-1}\{0\} = \{0\}$.

(b) φ (t) < t for all t > 0.

(c) $\lim_{r\to t+} \varphi$ (r) < t for all t > 0. Hence, it concluded that $\lim_{n\to\infty} \varphi^n(t) = 0$.

Theorem 2.14:

. Let (X, ≤) be a partially ordered set and suppose there is a G-metric G on X such that (X, G)is a complete G-metric space. Let F : X × X → X and g : X → X be such that F is continuous and has the mixed-g-monotone property. Assume there is a function $\varphi \in \Phi$ such that

$$G(F(x,y),F(u,v),F(w,z))\leq\frac{1}{2}\varphi(G(x,u,s) + G(y,v,t) -\varphi(\frac{\phi(G(gx,gu,gw) + G(gy,gv,gz)}{2})$$

(2.1)

for all x, y, u,v,s,t, ∈ X with gw ≼ gu ≼ gx and gy≼ gv ≼ gz. Suppose also that F(X×X) ⊆ g(X) and g is continuous and commutes with F If there exist x_0, y_0 ∈ X such that $gx_0 \preccurlyeq F(x_0, y_0)$ and $gy_0 \succcurlyeq F(y_0, x_0)$, then F and g have a coupled coincidence point, that is, there exists (x, y) ∈ X × X such that g(x) = F(x, y) and g(y) = F(y, x).

3.Main results

In this section, we give some fixed point theorems for compatible mappings in G-metric spaces. Our results extend some existing results in [3]. In [11],.

Definition 3.1.

The mapping F : X × X → X and g : X → X are said to be compatible if $\lim_{n\to\infty}G(gF(x_n,y_n),gF(x_n,y_n),F(gx_n,gy_n))=0$ and $\lim_{n\to\infty} G(gF(y_n, x_n), gF(y_n, x_n), F(gy_n, gx_n)) = 0$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n,y_n) = \lim_{n\to\infty} g(x_n) = x$, $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = y$ for all x, y ∈ X are satisfied.

Next, we prove our main results.

Theorem 3.2.

Let (X, ≼) be a partially ordered set and suppose there is a G-metric G on X such that (X, G) is a complete G-metric space. Let F : X × X → X and g : X → X be such that

$$\varphi(G(F(x,y),F(u,v)F(s,t)\leq$$

$$\frac{1}{2}\varphi\{max(G(gx, gu, gs)+ G(gy, gv, gt)-\varphi(max\frac{\psi (G(gx,g u,gs) + G(gy,gv,gt}{2})\}, \quad 3.1$$

for all x, y, u, v, s, t ∈ X with gx ≽ gu ≽ gs and gy ≼ gv≼ gt where either u ≠s or v ≠ t. Suppose F(XxX)⊆g(x)and g is continuous and compatible and F and also suppose either

(a).F is continuous or

(b) X has the following property:

I. if a nondecreasing sequence x_n is G-convergent to x, then $x_n \preccurlyeq x$, for all n,

II. if a nonincreasing sequence y_n is G-convergent to y, then $y_n \succcurlyeq y$, for all n.

If there exist x_0, y_0 ∈ X such that $x_0 \preccurlyeq F(x_0, y_0)$ and $y_0 \succcurlyeq F(y_0, x_0)$ then F has a coupled point; that is, there exist x, y ∈ X such that F(x, y) = gx and F(y, x) = gy.

Proof. Let x_0,y_0 ∈ X be such that $gx_0 \preccurlyeq F(x_0, y_0)$and $gy_0 \succcurlyeq F(y_0, x_0)$ since F(XxX) ⊆ g(x)

We can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$and $gy_1 = F(y_0, x_0)$. Again since $F(XxX) \subseteq g(x)$

We can choose $x_1, y_1 \in X$ such that $gx_2 = F(x_1, y_1)$and $gy_2 = F(y_1, x_1)$.

Since F has mixed G- monotone property

Write $gx_{n+1} = F(gx_n,gy_n)$, $gy_{n+1} = F(gy_n, gx_n)$ 3.2

for all $n \geq 1$. Due to the mixed monotone property of F, we can find

$gx_2 \succcurlyeq gx_1 \succcurlyeq gx_0$ and $gy_2 \preccurlyeq gy_1 \preccurlyeq gy_0$. By straightforward calculation, we obtain

$gx_0 \preccurlyeq gx_1 \preccurlyeq gx_2 \preccurlyeq \cdots \preccurlyeq gx_{n+1} \cdots$,

$gy_0 \succcurlyeq gy_1 \succcurlyeq g\, y_2 \succcurlyeq \cdots \succcurlyeq gy_{n+1} \cdots$. 3.3

Using 3.1 and 3.2, we obtain

$(G(x_{n+1}, x_{n+1}, x_n)=(\varphi(G(F(x_n,y_n),F(x_n, y_n)F(x_n, y_n)\leq$

$\varphi(G(gx_{n+1},gx_{n+1},gx_n)=(\varphi(G(F(gx_n,gy_n),F(gx_n,gy_n)F(gx_n,gy_n)\leq \frac{1}{2}\varphi\, max(G(gx_n,gx_n,gx_{n-1})+$

$G(gy_n,gy_n,gy_{n-1})-\varphi max(\frac{\psi\,(G(gx_n,gx_n,gx_{n-1})\, +\, G\,(gy_n,gy_n,gy_{n-1})}{2})$ 3.4

and similarly

$\varphi(G(gy_{n+1},gy_{n+1},gy_n)=(\varphi max(G(F(gy_n,gx_n),F(gy_n,gx_n)F(gy_n,gx_n))\leq$

$\frac{1}{2}\varphi max(G(gy_n,gy_n,gy_{n+1})+G(gx_n,gx_n,gx_{n+1})-\varphi max(\frac{\psi\,(G(gy_n,gy_n,gy_{n-1})\, +\, G\,(gx_n,gx_n,gx_{n-1})}{2})$. 3.5

Adding 3.4 and 3.5, we get

$\varphi max(G(gx_{n+1},gx_{n+1},gx_n)= \varphi max(G\,(gy_{n+1},gy_{n+1},gy_n)\leq \varphi(G(gx_n,gx_n,gx_{n-1})+ G(gy_n,gy_n,gy_{n-1})-$
$\varphi max(\frac{\psi\,(G(gx_n,gx_n,gx_{n-1})\, +\, G\,(gy_n,gy_n,gy_{n-1})}{2})$ 3.6

Using the property

$\Phi(t+s)\leq\phi(t)+\phi(s)$ for all $t, s \in [0, \infty)$, we get

$\varphi max(G\,(gx_{n+1},gx_{n+1},gx_n)+ \varphi max(G\,(gy_{n+1},gy_{n+1},gy_n)\leq \varphi max(G(gx_n,gx_n,gx_{n-1})+$
$G(gy_n,gy_n,gy_{n-1})- \varphi max(\frac{\psi\,(G(gx_n,gx_n,gx_{n-1})\, +\, G\,(gy_n,gy_n,gy_{n-1})}{2})$ 3.7

which implies that

$$\varphi max(G\,(gx_{n+1},gx_{n+1},gx_n)+ \varphi max(G\,(gy_{n+1},gy_{n+1},gy_n)\leq$$
$$\varphi max(G(gx_n, gx_n, gx_{n-1})\; +\; G\,(gy_n, gy_n, gy_{n-1})\;.\; 3.8$$

Since φ is nondecreasing, we have

$$G\,(gx_{n+1},gx_{n+1},gx_n)+ \varphi(G\,(gy_{n+1},gy_{n+1},gy_n)\leq$$

$(G(gx_n, gx_n, gx_{n-1})\; +\; G\,(gy_n, gy_n, gy_{n-1})$ 3.9

For all $n \in N$, set

$\{s_n\} = (G(gx_n, gx_n, gx_{n-1})\; +\; G\,(gy_n, gy_n, gy_{n-1})$ 3.10

Then a sequence sn is decreasing. Therefore, there exists some $s \geq 0$ such that

$\lim_{n\to\infty}\{s_n\} = \lim_{n\to\infty}\, \max\,\{(G(gx_n, gx_n, gx_{n-1}) +\; G\,(gy_n, gy_n, gy_{n-1})\}$ (3.11)

Now we have to show that $s =0$. On the contrary, suppose that $s > 0$. Letting $n \to \infty$ in 3.7 equivalently, sn is G-convergent to s and using the property of ϕ and ψ, we get

$\varphi(s) = \lim_{n\to\infty} \varphi\{s_n\}\leq \lim_{n\to\infty} max\{\, \varphi(s_n)\text{-}2\, \varphi\left\{\frac{s_{n-1}}{2}\right\}$

$\leq \varphi(s_n)\text{-}2\lim_{n\to\infty} \max\varphi(\frac{s_{n-1}}{2})$3.12

which is a contradiction. Thus s 0; from 3.11, we have

$\lim_{n \to \infty} \{s_n\} = \lim_{n \to \infty} \{(G(gx_n, gx_n, gx_{n-1}) + G(gy_n, gy_n, gy_{n-1})\}$ =0. 3.13

Again, we have to show that (x_n) and (y_n) are Cauchy sequences in the G-metric space (X, G) On the contrary, suppose that at least one of

$\{x_n\}$or $\{y_n\}$is not a Cauchy sequence in (X, G). Then there exists $\in > 0$, for which we can find subsequences $gx_{k(j)}$, $gx_{l(j)}$ and $gy_{l(j)}$, $gy_{l(j)}$ of the sequences$\{gx_n\}$and $\{gy_n\}$ respectively, with $k(j) > l(j) \geq j$, for all $j \in \mathbb{N}$ such that

$\alpha_j = G(gx_{k(j)}, gx_{k(j)},, gx_{l(j)}) + G(gy_{k(j)}, gy_{k(j)},, gy_{l(j)}) \geq \in$. 3.14

We may also assume that

$G(gx_{k(j)}, gx_{k(j)},, gx_{l(j)}) + G(gy_{k(j)}, gy_{k(j)},, gy_{l(j)}) < \in$, 3.15

$G(gx_{k(j)-1}, gx_{k(j)-1},, gx_{l(j)}) + G(gy_{k(j)-1}, gy_{k(j)-1},, gy_{lk(j)}) < \in$,

by choosing kj to be the smallest number exceeding lj, for which 3.14 holds. From 3.14 and 3.15 and using the rectangle inequality, we obtain $\in \leq g\alpha_j = G(gx_{k(j)}, gx_{k(j)},, gx_{l(j)}) + G(gy_{k(j)}, gy_{k(j)},, gy_{l(j)})$

$\leq G(gx_{k(j)}, gx_{k(j)},, gx_{l(j)}) + G(gx_{k(j)-1}, gx_{k(j)-1},, gx_{l(j)-1}) + G(gy_{k(j)}, gy_{k(j)},, gy_{l(j)}) + G(gy_{k(j)-1}, gy_{k(j)-1},, gy_{l(j)-1})$

$< G(gx_{k(j)}, gx_{k(j)},, gx_{l(j)}) + G(gy_{k(j)}, gy_{k(j)},, gy_{l(j)}) + \in$. 3.16

Letting $j \to \infty$ in the above inequality and using 3.13, we get

$\lim_{j\to\infty} g\alpha_j = \lim_{j\to\infty}\left[G\left(gx_{k(j)}, gx_{k(j)}, gx_{l(j)}\right) + G\left(gy_{k(j)}, gy_{k(j)}, gy_{l(j)}\right)\right] = \in$. 3.17 Again, by using rectangle inequality, we obtain

$g\alpha_j = G\left(gx_{k(j)}, gx_{k(j)}, gx_{l(j)}\right) + G\left(gy_{k(j)}, gy_{k(j)}, gy_{l(j)}\right)$

$\leq G(gx_{k(j)}, gx_{k(j)},, gx_{l(j)}) + G(gx_{k(j)+1}, gx_{k(j)+1}, gx_{l(j)+1}) + G(gx_{l(j)+1}, gx_{l(j)+1}, gx_{l(j)+1}) + G(gy_{k(j)}, gy_{k(j)},, gy_{l(j)}) + G(gy_{k(j)+1}, gy_{k(j)+1},, gy_{l(j)+1}) + G(gy_{l(j)+1}, gy_{l(j)+1},, gy_{l(j)+1})$

$= S_{l(j)} + G(gx_{k(j)}, gx_{k(j)},, gx_{l(j)}) + G(gx_{k(j)+1}, gx_{k(j)+1},, gx_{l(j)+1}) + G(gy_{k(j)}, gy_{k(j)},, gy_{l(j)}) + G(gy_{k(j)+1}, gy_{k(j)+1},, gy_{l(j)+1})$3.18

By using Lemma 2.4, the above inequality becomes

$g\alpha_i \leq S_{l(j)} + G(gx_{k(j)}, gx_{k(j)},, gx_{l(j)}) + G(gx_{k(j)+1}, gx_{k(j)+1},, gx_{l(j)+1}) + G(gy_{k(j)}, gy_{k(j)},, gy_{l(j)}) + G(gy_{k(j)+1}, gy_{k(j)+1},, gy_{l(j)+1})$; 3.19

this implies that

$g\alpha_i \leq S_{l(j)} + 2S_{k(j)} + G(gx_{k(j)+1}, gx_{k(j)+1},, gx_{l(j)+1}) + G(gy_{k(j)+1}, gy_{k(j)+1},, gy_{l(j)+1})$;. 3.20

Operating ϕ on both sides of the above inequality,

$\varphi(g\alpha_i) \leq \varphi(\left(S_{l(j)} + 2S_{k(j)}\right) + G(gx_{k(j)+1}, gx_{k(j)+1},, gx_{l(j)+1}) + G(gy_{k(j)+1}, gy_{k(j)+1},, gy_{l(j)+1}))$

$= \varphi(\left(S_{l(j)} + 2S_{k(j)}\right) + \varphi(G(gx_{k(j)+1}, gx_{k(j)+1},, gx_{l(j)+1}) + \varphi(G(gy_{k(j)+1}, gy_{k(j)+1},, gy_{l(j)+1}))$3.21

Now we find the expressions

$\varphi(G(gx_{k(j)+1}, gx_{k(j)+1},, gx_{l(j)+1})$and $\varphi(G(gy_{k(j)+1}, gy_{k(j)+1},, gy_{l(j)+1}))$ in terms of ϕ and ψ by using 3.1 and 3.2; that is,

$\varphi\max(G(gy_{k(j)+1}, gy_{k(j)+1},, gy_{l(j)+1})) = \varphi max(G(F(gx_{k(j)}, gy_{k(j)}), F(gx_{k(j)}, gy_{k(j)}) F(gx_{l(j)}, gy_{l(j)})$

$\leq \frac{1}{2} G(gx_{k(j)}, gx_{k(j)},, gx_{l(j)}) + G(gy_{k(j)}, gy_{k(j)},, gy_{l(j)})$-

$\varphi max(\frac{(G(gxk(j),gxk(j),,gxl(j)) + G(gyk(j),gyk(j),,gyl(j))}{2}$ 3.22

$\varphi\max(G(gy_{k(j)+1}, gy_{k(j)+1},, gy_{l(j)+1})) = \varphi(G(F(gy_{k(j)}, gx_{k(j)}), F(gy_{k(j)}, gx_{k(j)}) F(gy_{l(j)}, gx_{l(j)})$

$\leq \frac{1}{2}$ G ($gy_{k(j)}$, $gy_{k(j)}$,, $gy_{l(j)}$) + G ($gx_{k(j)}$, $gx_{k(j)}$, $gx_{l(j)}$)-

$\varphi max(\frac{(gy_{k(j)},gy_{k(j)}),,gy_{l(j)}) + G\ (gx_{k(j)},gx_{k(j)}),,gx_{l(j)}))}{2}$ 3.23

Adding 3.22 and 3.23, we get

φmax (G($gx_{k(j)+1}$, $gx_{k(j)+1}$,, $gx_{l(j)+1}$)+ φ max(G ($gy_{k(j)+1}$, $gy_{k(j)+1}$,, $gy_{l(j)+1}$)) $\leq \phi$ (α_j)$-$ 2ψ ($\frac{\alpha_j}{2}$).

3.24

From 3.21 and 3.24, we obtain

φmax $(g\alpha_j) \leq$

$\varphi max((\ S_{l(j)} +\ 2S_{k(j)}) + \varphi max(g\alpha_j) - 2\psi\ (\frac{\alpha_j}{2})$. . 3.25

Taking limit as j → ∞ on both sides of the above inequality, we get

$\varphi max(\in) \leq \varphi(0) +\ \varphi(\in) - 2 \lim_{j\to\infty} \psi\ (\frac{\alpha_j}{2})$

$= \varphi(\in) - 2 \lim_{\alpha_j\to\infty} \psi\ (\frac{\alpha_j}{2}) < \varphi(\in)$, 3.26

which is a contradiction, and hence(x_n) and (y_n) are Cauchy sequences in the G-metric space (X, G). Since (X, G) is complete G-metric space, hence { x_n }and { y_n }are G-convergent. Then, there exist x, y ∈ X such that (x_n) and (y_n) are G-convergent to x and y, respectively. Suppose that condition a holds. Letting n → ∞ in 3.2,
we get

gx = F (gx, gy) and gy = F(gy, gx) Lastly, suppose that assumption b holds. Since a sequence xn is nondecreasing and G-convergent to x and also (y_n) is nonincreasing sequence and G-convergent to y, by assumption(b),

we have $gx_n \succcurlyeq gx \succcurlyeq gx_0$ and $gy_n \preccurlyeq gy$ for all n. Using the rectangle inequality, write

G (gx,gx, F(gx, gy)) ≤ G (gx,gx, gx_{n+1}) + G (gx_{n+1} , gx_{n+1} F(gx, gy))

= G (gx,gx, gx_{n+1}) + G (F(gx_n, gx_n). F(gx_n, gx_n). F(gx_n, gx_n) F(gx, gy)). 3.27

Applying the function φ on both sides of the above equation and using 3.1, we have

φmax (G (gx,gx, F(gx, gy)) ≤ G (gx,gx, gx_{n+1}) + φ max(G (F(gx_n, gx_n). F(gx_n, gx_n). F(gx_n, gx_n) F(gx, gy))

≤ G (gx,gx, gx_{n+1}) + $\frac{1}{2}\varphi$ max(G (F(gx_n, gx_n). $\varphi max(\frac{\psi\ (G(gx_n,gx_n,gx) + G\ (gy_n,gy_n,y)}{2})$) . 3.28

Letting n → ∞, we get G (gx,gx, F(gx, gy)= 0.

Hence gx = F(gx,gy)

Similarly we obtain gy =F(gy, gx). Thus, we conclude that F has a coupled fixed point.

Corollary 3.2

Let (X, ≼) be a partially ordered set and suppose there is a G-metric G on X such that (X, G)is a complete G-metric space. Let F : X × X → X and g : X → X be such that

φ(G(F(x,y),F(u,v)F(s,t)≤

$\frac{1}{2}\varphi max$(G(gx, gu, gs)+ G(gy, gv, gt)$-\varphi$max {$\frac{\psi\ (G(gx,g\ u,gs) + G(\ gy,gv,gt}{2}$}), 3.1

for all x, y, u, v, s, t ∈ X with gx ≽ gu ≽ gs and gy ≼ gv≼ gt where either u ≠s or v ≠ t. Suppose F has a mixed monotone property and also suppose that either

(a).F is continuous or

(b) X has the following property:

I. if a nondecreasing sequence x_n is G-convergent to x, then $gx_n \preccurlyeq gx$, for all n,
II. if a nonincreasing sequence y_n is G-convergent to y, then $gy_n \succcurlyeq gy$, for all n.

If there exist $gx_0, gy_0 \in X$ such that $gx_0 \preccurlyeq F(gx_0, gy_0)$ and $y_0 \succcurlyeq F(gy_0, gx_0)$ then F has a coupled point; that is, there exist $x, y \in X$ such that $F(gx, gy) = gx$ and $F(gy, gx) = gy$.
. Proof. Taking $\varphi(t) = t$ in Theorem 3.1 and proceeding the same lines as in this theorem, we get the desired result.

Corollary 3.3.

Let $(X, \preccurlyeq)$ be a partially ordered set and suppose there is a G-metric G on X such that (X, G)is a complete G-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be such that

$\varphi(G(F(x,y),F(u,v)F(s,t) \leq$
$\frac{1}{2}\varphi(G(gx, gu, gs) + G(gy, gv, gt) - \varphi\frac{\psi\,(G(gx,gu,gs) + G(gy,gv,gt)}{2}),$ 3.1

for all $x, y, u, v, s, t \in X$ with $gx \succcurlyeq gu \succcurlyeq gs$ and $gy \preccurlyeq gv \preccurlyeq gt$ where either $u \neq s$ or $v \neq t$. Suppose F has a mixed monotone property and also suppose that either

(a).F is continuous or

(b) X has the following property:

I. if a nondecreasing sequence x_n is G-convergent to x, then $gx_n \preccurlyeq gx$, for all n,
II. if a nonincreasing sequence y_n is G-convergent to y, then $gy_n \succcurlyeq gy$, for all n.

If there exist $gx_0, gy_0 \in X$ such that $gx_0 \preccurlyeq F(gx_0, gy_0)$ and $y_0 \succcurlyeq F(gy_0, gx_0)$ then F has a coupled point; that is, there exist $x, y \in X$ such that $F(gx, gy) = gx$ and $F(gy, gx) = gy$.
. Proof. Taking $\phi(t) = t$ and $\psi(t) = (1 - k/2)t$ in Theorem 3.1 and proceeding the same lines as in this theorem, we get the desired result. **Remark 3.4.**

To assure the uniqueness of a coupled fixed point, we shall consider the following condition. If Y, is a partially ordered set, we endowed the product $Y \times Y$ with x, y u, v iff x u, y v,
3.31 for all $x, y, u, v \in Y \times Y$.

Theorem 3.5 Let $(X, \preccurlyeq)$ be a partially ordered set and suppose there is a G-metric G on X such that (X, G)is a complete G-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be such that

$\varphi(G(F(x,y),F(u,v)F(s,t) \leq$
$\frac{1}{2}\varphi(G(gx, gu, gs) + G(gy, gv, gt) - \varphi\frac{\psi\,(G(gx,gu,gs) + G(gy,gv,gt)}{2}),$ 3.1

for all $x, y, u, v, s, t \in X$ with $gx \succcurlyeq gu \succcurlyeq gs$ and $gy \preccurlyeq gv \preccurlyeq gt$ where either $u \neq s$ or $v \neq t$. Suppose F has a mixed monotone property and also suppose that either

(a).F is continuous or

(b) X has the following property:

I. if a nondecreasing sequence x_n is G-convergent to x, then $gx_n \preccurlyeq gx$, for all n,
II. if a nonincreasing sequence y_n is G-convergent to y, then $gy_n \succcurlyeq gy$, for all n.

If there exist $gx_0, gy_0 \in X$ such that $gx_0 \preccurlyeq F(gx_0, gy_0)$ and $y_0 \succcurlyeq F(gy_0, gx_0)$ then F has a coupled point; that is, there exist $x, y \in X$ such that $F(gx, gy) = gx$ and $F(gy, gx) = gy$.

Proof. It follows from Theorem 3.1 that the set of coupled fixed points is nonempty. Suppose (x, y) and (s, t) are coupled fixed points of the mappings $F : X \times X \to X$; that is, F(gx, gy) = gx and F(gy, gx) = gy and gs = F(gs, gt), gt = F(t, s). By assumption there exists u, v in X × X such that
(F(gu, gv), F(gv,gu)is comparable to F(gx, gy), F(gy, gx)and F(gs, gt),F(gt, gs).
Put
gu = gu_0 and gv = gv_0 and choose gu_1 and $gv_1 \in X$ such that
$F(gu_1, gv_1) = gu_1$ and $F(gv_1, gu_1) = gv_1$ Thus, we can define two sequences gu_n and gv_n as
$F(gu_n, gv_n) = gu_{n+1}$)and $F(gv_n, gu_n) = gv_{n+1}$,
3.32
Since (gu,gv)is comparable to (gx, gy).
we can assume that(gx, gy).$\succcurlyeq$ (gu, gv) = (gu_0, gv_0). Then it is easy to show that $(gu_n, gv_n)$and (gx, gy)are comparable; that is, (gx, gy).$\succcurlyeq (gu_n, gv_n)$for all n. Thus, from 3.1, we have

$\varphi(G(gu_{n+1}, gx, gx) = \varphi\, G(F(gu_n, gv_n), F(gx, gy), F(gx, gy) \leq$
$\frac{1}{2}\varphi(G(gu_n, gy, gy, gu_{n+1}) + G(gv_n, gx, gx))) - \varphi\frac{\psi\,(G(gu_n, g\,x, gx) + G(\,gv_n gy, gy)}{2})$ 3.33
$\varphi(G(gy, gy\, gv_{n+1}) = \varphi\, G(F(gy, gx), F(gy, gx)\, F(gv_n, gu_n),) \leq$
$\frac{1}{2}\varphi(G(gy, gy\, gv_n) + G(gx, gx, gv_n,))) - \varphi\frac{\psi\,(G(gy, gy\, gv\;\;) + G(gx, gx, gu_n,)}{2})$. 3.34

Using the property of φ and adding 3.33 and 3.34, we get
$\varphi(G(gu_{n+1}, gx, gx) + G(gv_{n+1}, gy, gy) = \varphi\, G(F(gu_n, gv_n \leq$
$\varphi(G(gu_{n+1}, gx, gx) + G(gv_{n+1}, gy, gy))) - \varphi$ (3.36)

$$G(gu_{n+1}\, gx, gx) + G(gv_{n+1}\, gy, gy) \leq G(gu_{n+1}\, gx, gx) + G(gv_{n+1}\, gy, gy)$$

We see that the sequence $G(gu_n\, gx, gx) + G\big(gv_{n,}\, gy, gy\big) \leq$ is decreasing; there exists some $\xi \geq 0$ such that

$\lim_{n\to\infty} G(gu_n\, gx, gx) + G\big(gv_{n,}\, gy, gy\big) \leq \xi$. 3.38

Now we have to show that $\xi = 0$. On the contrary, suppose that $\xi > 0$. Letting $n \to \infty$ in 3.35, we get

$\varphi(g\xi) \leq \varphi(g\xi) - 2\psi \lim_{n\to\infty}(\frac{G(gu_n\, gx, gx) + G(gv_{n,}\, gy, gy)}{2}) < \varphi(g\xi)$, 3.39

which is not possible. Hence $\xi = 0$. Therefore, 3.38 becomes

$\lim_{n\to\infty} G(gu_n\, gx, gx) + G\big(gv_{n,}\, gy, gy\big) = 0\ \ \xi$

$\lim_{n\to\infty} G(gu_n\, gx, gx) = 0 = G\big(gv_{n,}\, gy, gy\big)$

which implies

$\lim_{n\to\infty} G(gu_n\, gx, gx) = 0 = G\big(gv_{n,}\, gy, gy\big)$. 3.41

Similarly, we can show that $\lim_{n\to\infty} G(gu_n\, gs, gs) = 0 = G\big(gv_{n,}\, gt, gt\big)$ We conclude that gx =gs and gy =gt. Thus, F has a unique coupled fixed point.

References :

1. M. Abbas, M. A. Khan, S. Radenovi´c, Common coupled fixed point theorem in cone metric space for w ∗ -compatible mappings, Appl. Math. Comput., 217, (2010), 195–202. 1
2. H. Aydi, B. Damjanovi´c, B. Samet, W. Shatanawi, Coupled fixed point theorems for nonlinear contractions in partially ordered G-metric spaces, Math. Comput. Modelling, 54 (2011), 2443–2450.
3. T. G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal., 65 (2006), 1379–1393. 1, 1.11, 1.12
4. B. S. Choudhury, A. Kundu, A coupled coincidence point result in partially ordered metric spaces for compatible mappings, Nonlinear Anal., 73 (2010), 2524–2531. 1
5. R. Chugh, T. Kadian, A. Rani, B. E. Rhoades, Property P in G-metric spaces, Fixed Point Theory Appl., 2010 (2010),
6. Lj. Ciri´c, N. Caki´c, M. Rajovi´c, J. S. Ume, ´ Monotone generalized nonlinear contractions in partially ordered metric spaces, Fixed Point Theory Appl., 2008 (2008),
7. Lj. Ciri´c, D. Mihet, R. Saadati, ´ Monotone generalized contractions in partially ordered probabilistic metric spaces, Topology Appl., 156(17) (2009), 2838–2844.
8. M. Jain, K. Tas, B. E. Rhoades, N. Gupta, Coupled fixed point theorems for generalized symmetric contractions in partially ordered metric spaces and applications, J. Comput. Anal. Appl., 16 (2014), 438–454. 1
9. E. Karapinar, P. Kumam, I. Merhan, Coupled fixed point theorems on partially ordered G-metric spaces, Fixed Point Theory Appl., 2012 (2012),
10. V. Lakshmikantham, Lj. Ciri´c, ´ Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal., 70 (2009), 4341–4349.
11. N. V. Luong, N. X. Thuan, Coupled fixed point theorems in partially ordered G-metric spaces, Math. Comput. Modelling 55, (2012), 1601–1609.
12. S. A. Mohiuddine, A. Alotaibi, Coupled coincidence point theorems for compatible mappings in partially ordered intuitionistic generalized fuzzy metric spaces, Fixed Point Theory Appl., 2013 (2013),
13. Z. Mustafa, W. Hatanawi, M. Bataineh, Existence of fixed point results in G-metric spaces, Int. J. Math. Math. Sci., 2009 (2009),
14. H. K. Nashine, B. Samet, Fixed point results for mappings satisfying (ψ, ϕ)-weakly contractive condition in partially ordered metric spaces, Nonlinear Anal., 74 (2011), 2201–22209.
15. B. Samet, Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces, Nonlinear Anal., 72, (2010), 4508–4517. 1